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GROUP REPRESENTATIONS ON RIEMANN-ROCH SPACES OF SOME HURWITZ CURVES

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Abstract. Let $q > 1$ denote an integer relatively prime to $2, 3, 7$ and for which $G = PSL(2, q)$ is a Hurwitz group for a smooth projective curve X defined over $\mathbb{C}$. We compute the G-module structure of the Riemann-Roch space $L(D)$, where D is an invariant divisor on X of positive degree. This depends on a computation of the ramification module, which we give explicitly. In particular, we obtain the decomposition of $H^1(X, \mathbb{C})$ as a G-module.

1. Introduction

Let X be a smooth projective curve over an algebraically closed field k, and let $k(X)$ denote the function field of X (the field of rational functions on X). If D is any divisor on X then the Riemann-Roch space $L(D)$ is a finite dimensional k-vector space given by

$$L(D) = L_X(D) = \{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\},$$

where $\mathrm{div}(f)$ denotes the (principal) divisor of the function $f \in k(X)$. If G is a finite group of automorphisms of X, then G has a natural action on $k(X)$, and on the group $\mathrm{Div}(X)$ of divisors on X. If D is a G-invariant divisor, then G also acts on the vector space $L(D)$, making it into a $k[G]$-module.

The problem of finding the $k[G]$-module structure of $L(D)$ was first considered in the case where $k = \mathbb{C}$ and D is canonical, i.e. $L(D)$ is the space of holomorphic differentials on X. This problem was solved by Hurwitz for G cyclic, and then by Chevalley and Weil for general G. More generally, the problem has been solved by work of Ellingsrud and Lønsted [EL], Kani [K], Nakajima [N], and Borne [B]. This has resulted in the following equivariant Riemann-Roch formula for the class of $L(D)$ (denoted by square brackets) in the Grothendieck group $R_k(G)$, in the case where D is non-special:

$$[L(D)] = (1 - g_{X/G})[k[G]] + [\deg_{eq}(D)] - [\tilde{\Gamma}_G]. \tag{1}$$

Here $g_{X/G}$ is the genus of X/G, $\deg_{eq}(D)$ is the equivariant degree of D, and $\tilde{\Gamma}_G$ is the (reduced) ramification module (this notation will be defined in sections 4.1 and 4.2).

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Explicitly computing the $k[G]$-module structure of $L(D)$ in specific cases is of interest currently due to advances in the theory of algebraic-geometric codes. Permutation decoding algorithms use this information to increase their efficiency.

In this paper, we consider the case where X is a Hurwitz curve with automorphism group $G = PSL(2, q)$ for some prime power q, over $k = \mathbb{C}$. Using the equivariant Riemann-Roch formula above (1) and the representation theory of $PSL(2, q)$, we compute explicitly the $\mathbb{C}[G]$-module structure of $L(D)$ for a general invariant effective divisor D. In the case where D is a canonical divisor, this yields an explicit computation for the $\mathbb{C}[G]$-module structure of $H^1(X, \mathbb{C})$.

We are also interested in rationality questions. We find that $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure, and therefore may be computed more simply (see Joyner and Ksir [JK1]), as follows:

$$\tilde{\Gamma}_G = \bigoplus_{\pi \in G^*} \left[\sum_{\ell=1}^{L} (\dim \pi - \dim (\pi^{H_\ell})) \frac{R_\ell}{2} \right] \pi. \tag{2}$$

The sum is over all conjugacy classes of cyclic subgroups of G, H_ℓ is a representative cyclic subgroup, π^{H_ℓ} indicates the fixed part of π under the action of H_ℓ, and R_ℓ denotes the number of branch points in Y over which the decomposition group is conjugate to H_ℓ. For some but not all divisors D, $L(D)$ has a $\mathbb{Q}[G]$-module structure, and may also be computed more simply.

The organization of this paper is as follows. In section 2, we recall some facts about Hurwitz curves and Hurwitz groups. In section 3, we review the representation theory of $PSL(2, q)$, and compute the induced characters necessary for the following section. Our main results are in section 4, where we compute the ramification module, the equivariant degree for any invariant divisor D, and thus the structure of $L(D)$. At the end of section 4 we compute the $\mathbb{C}[G]$-module structure of $H^1(X, \mathbb{C})$. In section 5, we discuss rationality questions, using the results of [JK1] to give more streamlined formulas for the ramification module, and in some cases for $L(D)$.

2. Hurwitz curves

The automorphism group G of a smooth projective curve of genus $g > 1$ over an algebraically closed field k of characteristic zero satisfies the *Hurwitz bound*

$$|G| \leq 84 \cdot (g - 1).$$

A curve which attains this bound is called a *Hurwitz curve* and its automorphism group is called a *Hurwitz group.*

2.1. Classification.

The number of distinct Hurwitz groups is infinite, and to each one corresponds a finite number of Hurwitz curves. Nevertheless, these curves are quite rare; in particular, the Hurwitz genus values are known to form a rather sparse set of positive integers (see Larsen [L]).

Hurwitz groups are precisely those groups which occur as non-trivial finite homomorphic images of the 2,3,7-triangle group

$$\Delta = \langle a, b : a^2 = b^3 = (ab)^7 = 1 \rangle.$$

This is most naturally viewed as the group of orientation-preserving symmetries of the tiling of the hyperbolic plane $\mathbf{H}$ generated by reflections in the sides of a fundamental triangle having angles $\pi/2$, $\pi/3$, and $\pi/7$. Each proper normal finite-index subgroup $K \triangleleft \Delta$ corresponds to a Hurwitz group $G = \Delta/K$. The associated Hurwitz curve now appears (with $k = \mathbb{C}$) as a compact hyperbolic surface $\mathbf{H}/K$ regularly tiled by a finite number of copies of the fundamental triangle. G is the group of orientation-preserving symmetries of this tiling, with fundamental domain consisting of one fundamental triangle plus one reflected triangle. (From this perspective, the Hurwitz bound simply says that there is no smaller polygon which gives a regular tiling of $\mathbf{H}$.)

We note that Δ has only a small number of torsion elements (up to conjugacy). These are the non-trivial powers of a, b, and ab. Each acts as a rotation of order 2, 3, or 7, and has as its fixed point one vertex of (some copy of) the fundamental triangle. Clearly no other point of the tiling can occur as a fixed point; this is true both for the tiling of $\mathbf{H}$ and the induced tilings on the quotient surfaces. In other words, all points *other* than the tiling vertices have trivial stabilizer.

It follows easily from the above presentation for Δ that a group is Hurwitz if and only if it is generated by two elements having orders 2 and 3, and whose product has order 7. This characterization has made possible much of the work in classifying Hurwitz groups. The most relevant for our investigation is the following result of Macbeath (see [M]):

> The simple group $PSL(2, q)$ is Hurwitz in exactly three cases:
> i) $q = 7$;
> ii) q is prime, with $q \equiv \pm 1 \pmod 7$;
> iii) $q = p^3$, with p prime and $p \equiv \pm 2, \pm 3 \pmod 7$.

In particular, $PSL(2, 8)$ and $PSL(2, 27)$ are Hurwitz groups. We shall require that q be relatively prime to $2 \cdot 3 \cdot 7$, but this excludes just three possibilities, namely $q \in \{7, 8, 27\}$. Note that in all of the cases we consider, $q \equiv \pm 1 \pmod 7$.

The order of $PSL(2, q)$ (for odd q) is $q(q^2 - 1)/2$. Hence we obtain

$$g = 1 + \frac{q(q^2 - 1)}{168}$$

as the genus of the corresponding curve(s).

For completeness, we remark that there are three distinct Hurwitz curves when q is prime (apart from $q = 7$), and just one when $q = p^3$. However, this has no bearing on the representations that we study.

In addition, there are other known families of Hurwitz groups. For example, all Ree groups are Hurwitz, as are all but finitely many of the alternating groups. See Conder [C] for a summary of such results.

2.2. **Ramification data.** Let X be a Hurwitz curve with automorphism group G and let

$$\psi : X \to Y = X/G \tag{3}$$

denote the quotient map. By again viewing X as a hyperbolic surface, the ramification data are easily deduced. The quotient Y is formed by one fundamental triangle and its mirror image, with the natural identifications on their boundaries. Hence it is a surface of genus 0 with 3 metric singularities. Thus ψ has exactly three branch points. The stabilizer subgroups of the corresponding ramification points in X are cyclic, of orders 2, 3, and 7. We label the three branch points P_1, P_2, and P_3, so that if $P \in \psi^{-1}(P_1)$, then P has stabilizer subgroup of order 2, if $P \in \psi^{-1}(P_2)$, P

has stabilizer subgroup of order 3, and if $P \in \psi^{-1}(P_3)$, P has stabilizer subgroup of order 7.

3. Representation theory of $PSL(2,q)$

3.1. **General theory on representations of PSL(2,q).** We first review the representation theory of $G = PSL(2,q)$ over $\mathbb{C}$, following the treatment in [FH], to fix notation.

Let $\mathbb{F} = GF(q)$ be the field with q elements. The group $PSL(2,q)$ has $3+(q-1)/2$ conjugacy classes of elements. Let $\varepsilon \in \mathbb{F}$ be a generator for the cyclic group $\mathbb{F}^\times$. Then each conjugacy class will have a representative of exactly one of the following forms:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}. \tag{4}$$

The irreducible representations of $PSL(2,q)$ include the trivial representation $\mathbf{1}$ and one irreducible V of dimension q. All but two of the others fall into two types: representations W_α of dimension $q+1$ ("principal series"), and X_β of dimension $q-1$ ("discrete series"). The principal series representations W_α are indexed by homomorphisms $\alpha : \mathbb{F}^\times \to \mathbb{C}^\times$ with $\alpha(-1) = 1$. The discrete series representations X_β are indexed by homomorphisms $\beta : T \to \mathbb{C}^\times$ with $\beta(-1) = 1$, where T is a cyclic subgroup of order $q+1$ of $\mathbb{F}(\sqrt{\varepsilon})^\times$. The characters of these are as follows:

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$
$\mathbf{1}$	1	1	1	1	1
X_β	$q-1$	0	-1	-1	$-\beta(x+\sqrt{\varepsilon}y) - \beta(x-\sqrt{\varepsilon}y)$
V	q	1	0	0	-1
W_α	$q+1$	$\alpha(x)+\alpha(x^{-1})$	1	1	0

Let ζ be a primitive qth root of unity in $\mathbb{C}$. Let ξ and ξ' be defined by

$$\xi = \sum_{\left(\frac{a}{q}\right)=1} \zeta^a \text{ and } \xi' = \sum_{\left(\frac{a}{q}\right)=-1} \zeta^a, \tag{5}$$

where the sums are over the quadratic residues and nonresidues $\pmod q$, respectively. If $q \equiv 1 \bmod 4$, then the principal series representation W_{α_0} corresponding to

$$\begin{array}{rcl} \alpha_0 : \mathbb{F}^\times & \to & \mathbb{C}^\times \\ \varepsilon & \mapsto & -1 \end{array}$$

is not irreducible, but splits into two irreducibles W' and W'', each of dimension $(q+1)/2$. Their characters satisfy:

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$
W'	$\frac{q+1}{2}$	$\alpha_0(x)$	$1+\xi$	$1+\xi'$	0
W''	$\frac{q+1}{2}$	$\alpha_0(x)$	$1+\xi'$	$1+\xi$	0

Let τ denote a generator of T. Similarly, if $q \equiv 3 \bmod 4$, then the discrete series representation X_{β_0} corresponding to

$$\begin{array}{rcl} \beta_0 : T & \to & \mathbb{C}^\times \\ \tau & \mapsto & -1 \end{array}$$

splits into two irreducibles X' and X'', each of dimension $(q-1)/2$. Their characters satisfy:

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$
X'	$\frac{q-1}{2}$	0	ξ	ξ'	$-\beta_0(x+y\sqrt{\varepsilon})$
X''	$\frac{q-1}{2}$	0	ξ'	ξ	$-\beta_0(x+y\sqrt{\varepsilon})$

According to Janusz [Ja], the Schur index of each irreducible representation of G is 1.

There is a "Galois action" on the set of equivalence classes of irreducible representations of G as follows. Let χ denote an irreducible character. The character values $\chi(g)$ lie in $\mathbb{Q}(\mu)$, where μ is a primitive m^{th} root of unity and $m = q(q^2-1)/4$. Let $\mathcal{G} = Gal(\mathbb{Q}(\mu)/\mathbb{Q})$ denote the Galois group. For each integer j relatively prime to m, there is an element σ_j of $\mathcal{G}$ taking μ to μ^j. This Galois group element will act on representations by taking a representation with character values $(a_1, \dots, a_n)$ to a representation with character values $(\sigma_j(a_1), \dots, \sigma_j(a_n))$. Representations with rational character values will be fixed under this action. Because the Schur index of each representation is 1, representations with rational character values will be defined over $\mathbb{Q}$.

The action of the Galois group $\mathcal{G}$ can easily be seen from the character table. It will fix the trivial representation and the q-dimensional representation V. Its action permutes the set of $q-1$-dimensional "principal series" representations X_β, and the set of $q+1$-dimensional "discrete series" representations W_α. In the case $q \equiv 1 \pmod 4$, the Galois group will exchange the two $(q+1)/2$-dimensional representations W' and W''; if $q \equiv 3 \pmod 4$, the Galois group will exchange the two $(q-1)/2$-dimensional representations X' and X''.

3.2. **Induced characters.** We will be interested in the induced characters from subgroups of orders 2, 3, and 7. For each value of q, each of these subgroups is unique up to conjugacy; we can choose subgroups H_2 of order 2, H_3 of order 3, and H_7 of order 7 that are generated by elements of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}.$$

Which of these two forms each generator will take depends on q mod 4, mod 3, and mod 7, respectively. Recall that we defined generators ε of the cyclic group $\mathbb{F}^\times$, of order $q-1$, and τ of the cyclic group $T \subseteq \mathbb{F}(\sqrt{\varepsilon})^\times$ of order $q+1$, respectively. We define numbers i, ω, and ϕ to be primitive roots of unity as follows.

When $q \equiv 1 \pmod 4$, let i denote an element in $\mathbb{F}^\times$ whose square is -1 (one can take $i = \varepsilon^{(q-1)/4}$). Then the subgroup H_2 of order 2 in $PSL(2,q)$ is generated by

$$\begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}.$$

If $q \equiv 3 \pmod 4$, then we take $i = x_i + \sqrt{\varepsilon} y_i$ to be an element of T whose square is -1 (one can take $i = \tau^{(q+1)/4}$). Then the subgroup H_2 of order 2 in $PSL(2,q)$ is generated by

$$\begin{pmatrix} x_i & \varepsilon y_i \\ y_i & x_i \end{pmatrix}.$$

Similarly, we define ω to be a primitive 6th root of unity. In the case where $q \equiv 1 \pmod 3$, we can take $\omega = \varepsilon^{(q-1)/6} \in \mathbb{F}^\times$. When $q \equiv -1 \pmod 3$, we take $\omega = x_\omega + \sqrt{\varepsilon} y_\omega = \tau^{(q+1)/6} \in T$. The subgroup H_3 of order 3 in $PSL(2,q)$ will then be generated by

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \text{ if } q \equiv 1 \pmod 3, \text{ or } \begin{pmatrix} x_\omega & \varepsilon y_\omega \\ y_\omega & x_\omega \end{pmatrix}, \text{ if } q \equiv -1 \pmod 3.$$

Lastly, we want to define ϕ to be a primitive 14th root of unity. Recall that $q \equiv \pm 1 \pmod 7$. If $q \equiv 1 \pmod 7$, then we can take $\phi = \varepsilon^{(q-1)/14} \in \mathbb{F}^\times$, and if $q \equiv -1 \pmod 7$, then we can take $\phi = x_\phi + \sqrt{\varepsilon} y_\phi = \tau^{(q+1)/14} \in T$. The subgroup H_7 of order 7 in $PSL(2,q)$ will then be generated by

$$\begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}, \ q \equiv 1 \pmod 3, \text{ or } \begin{pmatrix} x_\phi & \varepsilon y_\phi \\ y_\phi & x_\phi \end{pmatrix}, \ q \equiv -1 \pmod 3.$$

With these definitions, it is easy to compute the restrictions of the irreducible representations of $PSL(2,q)$ to the subgroups above. We omit the details, but the computations for the groups of order 2 and 3 are given in [JK2], and the computation for the group of order 7 is very similar. Using Frobenius reciprocity, we then obtain the corresponding induced representations. In each case, we denote a primitive character of the cyclic group H_k by θ_k.

3.2.1. *Induced characters from H_2.* The induced representations from the nontrivial character of H_2 are given below. The multiplicities depend on $q \pmod 8$. Note that most representation have the same multiplicity as V. When $i \in \mathbb{F}^\times$, i.e. when $q \equiv 1 \pmod 4$, the multiplicity of a discrete series representation W_α depends on the sign of $\alpha(i)$. Recall that $\alpha(-1) = 1$, so $\alpha(i) = \pm 1$. the multiplicity of W_α will be the same as the multiplicity of V if $\alpha(i) = 1$ and one larger if $\alpha(i) = -1$. Similarly, when $q \equiv 3 \pmod 4$ and $i \in T$, the multiplicity of a principal series representation X_β depends on the sign of $\beta(i)$. In this case the multiplicity of X_β will be the same as the multiplicity of V when $\beta(i) = 1$, and one less if $\beta(i) = -1$. Lastly, the signs of $\alpha_0(i)$ or $\beta_0(i)$ depend on $q \pmod 8$ and determine the multiplicities of W' and W'' or X' and X'', respectively. A similar pattern will hold for the induced representations from H_3 and H_7.

For $q \equiv 1 \pmod 8$,

$$Ind_{H_2}^G \theta_2 = \frac{q-1}{2}\left[\frac{1}{2}(W' + W'') + \sum_{\beta} X_\beta + V + \sum_{\alpha(i)=1} W_\alpha\right] + \frac{q+3}{2}\sum_{\alpha(i)=-1} W_\alpha.$$

For $q \equiv 3 \pmod 8$,

$$Ind_{H_2}^G \theta_2 = \frac{q+1}{2}\left[\sum_{\beta(i)=1} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \frac{q-3}{2}\left[\frac{1}{2}(X' + X'') + \sum_{\beta(i)=-1} X_\beta\right].$$

For $q \equiv 5 \pmod 8$,

$$Ind_{H_2}^G \theta_2 = \frac{q-1}{2}\left[\sum_{\beta} X_\beta + V + \sum_{\alpha(i)=1} W_\alpha\right] + \frac{q+3}{2}\left[\frac{1}{2}(W' + W'') + \sum_{\alpha(i)=-1} W_\alpha\right].$$

And for $q \equiv 7 \pmod 8$,

$$Ind_{H_2}^G \theta_2 = \frac{q+1}{2}\left[\frac{1}{2}(X' + X'') + \sum_{\beta(i)=1} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \frac{q-3}{2}\sum_{\beta(i)=-1} X_\beta.$$

3.2.2. *Induced characters from H_3.* The induced representations from the two non-trivial characters θ_3 and θ_3^2 of H_3 are the same. In this case the multiplicities depend on $q \pmod{12}$, which determines whether the 6th root of unity ω is in $\mathbb{F}^\times$, or in $T \subset \mathbb{F}(\sqrt{\varepsilon})^\times$. Now the multiplicity of a discrete (resp. principal) series representation W_α (resp. X_β) will be the same as the multiplicity of V if $\alpha(\phi) = 1$ (resp. $\beta(\phi) = 1$) and one larger (resp. smaller) if $\alpha(\phi) = e^{\frac{\pm 2\pi i}{3}}$ (resp. $\beta(\phi) = e^{\frac{\pm 2\pi i}{3}}$). The signs of $\alpha_0(\omega)$ or $\beta_0(\omega)$ depend on $q \pmod{12}$ and determine the multiplicities of W' and W'' or X' and X'', respectively.

If $q \equiv 1 \pmod{12}$, we have

$$Ind_{H_3}^G \theta_3 = \frac{q-1}{3}\left[\frac{1}{2}(W' + W'') + \sum_{\beta} X_\beta + V + \sum_{\alpha(\omega)=1} W_\alpha\right] + \frac{q+2}{3}\sum_{\alpha(\omega)=e^{\frac{\pm 2\pi i}{3}}} W_\alpha.$$

If $q \equiv 5 \pmod{12}$, we have

$$Ind_{H_3}^G \theta_3 = \frac{q+1}{3}\left[\frac{1}{2}(W' + W'') + \sum_{\beta(\omega)=1} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \frac{q-2}{3}\sum_{\beta(\omega)=1} X_\beta.$$

If $q \equiv 7 \pmod{12}$, we have

$$Ind_{H_3}^G \theta_3 = \frac{q-1}{3}\left[\frac{1}{2}(X' + X'') + \sum_{\beta} X_\beta + V + \sum_{\alpha(\omega)=1} W_\alpha\right] + \frac{q+2}{3}\sum_{\alpha(\omega)=e^{\frac{\pm 2\pi i}{3}}} W_\alpha.$$

And if $q \equiv 11 \pmod{12}$, we have

$$Ind_{H_3}^G \theta_3 = \frac{q+1}{3}\left[\frac{1}{2}(X' + X'') + \sum_{\beta(\omega)=1} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \frac{q-2}{3}\sum_{\beta(\omega)=e^{\frac{\pm 2\pi i}{3}}} X_\beta.$$

3.2.3. *Induced characters from H_7.* For H_7, the induced representations from the six nontrivial characters θ_7^k are not all the same, but depend on k. These representations also depend on $q \pmod{28}$, which determines whether the 14th root of unity ϕ is in $\mathbb{F}^\times$ or $\mathbb{F}(\sqrt{\varepsilon})^\times$. For an induced nontrivial character $\mathrm{Ind}_{H_7}^G \theta_7^k$, the multiplicity of a discrete (resp. principal) series representation W_α (resp. X_β) will be the same as the multiplicity of V if $\alpha(\phi) \neq e^{\pm\frac{2\pi ik}{7}}$ (resp. $\beta(\phi) \neq e^{\pm\frac{2\pi ik}{7}}$) and one larger (resp. smaller) if $\alpha(\phi) = e^{\pm\frac{2\pi ik}{7}}$ (resp. $\beta(\phi) = e^{\pm\frac{2\pi ik}{7}}$). The signs of $\alpha_0(\phi)$ or $\beta_0(\phi)$ depend on $q \pmod{28}$ and determine the multiplicities of W' and W'' or X' and X'', respectively.

If $q \equiv 1 \pmod{28}$, we have

$$Ind_{H_7}^G \theta_7^k = \frac{q-1}{7}\left[\frac{1}{2}(W'+W'') + \sum_\beta X_\beta + V + \sum_{\alpha(\phi)\neq e^{\pm\frac{2\pi ik}{7}}} W_\alpha\right] + \frac{q+6}{7}\sum_{\alpha(\phi)=e^{\pm\frac{2\pi ik}{7}}} W_\alpha.$$

If $q \equiv 13 \pmod{28}$, we have

$$Ind_{H_7}^G \theta_7^k = \frac{q+1}{7}\left[\frac{1}{2}(W'+W'') + \sum_{\beta(\phi)\neq e^{\pm\frac{2\pi ik}{7}}} X_\beta + V + \sum_\alpha W_\alpha\right] + \frac{q-6}{7}\sum_{\beta(\phi)=e^{\pm\frac{2\pi ik}{7}}} X_\beta.$$

If $q \equiv 15 \pmod{28}$, we have

$$Ind_{H_7}^G \theta_7^k = \frac{q-1}{7}\left[\frac{1}{2}(X'+X'') + \sum_\beta X_\beta + V + \sum_{\alpha(\phi)\neq e^{\pm\frac{2\pi ik}{7}}} W_\alpha\right] + \frac{q+6}{7}\sum_{\alpha(\phi)=e^{\pm\frac{2\pi ik}{7}}} W_\alpha.$$

And if $q \equiv 27 \pmod{28}$, we have

$$Ind_{H_7}^G \theta_7^k = \frac{q+1}{7}\left[\frac{1}{2}(X'+X'') + \sum_{\beta(\phi)\neq e^{\pm\frac{2\pi ik}{7}}} X_\beta + V + \sum_\alpha W_\alpha\right] + \frac{q-6}{7}\sum_{\beta(\phi)=e^{\pm\frac{2\pi ik}{7}}} X_\beta.$$

4. The Riemann-Roch space as a G-module

Now we have all of the pieces we need to compute the G-module structure of the Riemann-Roch space $L(D)$ of a general G-invariant divisor D. We will first compute the ramification module, which does not depend on D. We will then compute the equivariant degree of D, and use the equivariant Riemann-Roch formula (1) to compute $L(D)$.

4.1. **Ramification module.** The ramification module introduced by Kani [K] and Nakajima [N] is defined by

$$\Gamma_G = \sum_{P\in X_{\mathrm{ram}}} \mathrm{Ind}_{G_P}^G \left(\sum_{\ell=1}^{e_P-1} \ell\theta_P^\ell\right),$$

where the first sum is over the ramification points of $\psi : X \to Y = X/G$, and θ_P is the ramification character at a point P. Both Kani and Nakajima showed that there is a G-module $\tilde{\Gamma}_G$ such that $\Gamma_G \simeq \bigoplus_{|G|} \tilde{\Gamma}_G$. Because Γ_G does not figure in our calculations, we abuse notation and refer to $\tilde{\Gamma}_G$ as the *ramification module*.

Recall from section 2.2 that $\psi : X \to Y = X/G$ has three branch points, P_1, P_2, and P_3. If $P \in \psi^{-1}(P_1)$, G_P has order 2, so there are $\frac{|G|}{2}$ ramification points where G_P is conjugate to H_2. If $P \in \psi^{-1}(P_2)$, G_P has order 3, so there are $\frac{|G|}{3}$ ramification points where G_P is conjugate to H_3, and if $P \in \psi^{-1}(P_3)$, G_P has order 7, so there are $\frac{|G|}{7}$ ramification points where G_P is conjugate to H_7. Thus

$$\tilde{\Gamma}_G = \frac{1}{|G|}\left(\frac{|G|}{2}\operatorname{Ind}_{H_2}^G \theta_2 + \frac{|G|}{3}\sum_{\ell=1}^{2} \ell \operatorname{Ind}_{H_3}^G \theta_3^\ell + \frac{|G|}{7}\sum_{\ell=1}^{6} \ell \operatorname{Ind}_{H_7}^G \theta_7^\ell\right). \tag{6}$$

To compute this, we break it into three pieces:

$$\begin{aligned}
\tilde{\Gamma}_G &= \Gamma_{H_2} + \Gamma_{H_3} + \Gamma_{H_7},\\
\Gamma_{H_2} &= \frac{1}{2}\operatorname{Ind}_{H_2}^G \theta_2,\\
\Gamma_{H_3} &= \frac{1}{3}(\operatorname{Ind}_{H_3}^G \theta_3 + 2\operatorname{Ind}_{H_3}^G \theta_3^2),\\
\Gamma_{H_7} &= \frac{1}{7}(\operatorname{Ind}_{H_7}^G \theta_7 + 2\operatorname{Ind}_{H_7}^G \theta_7^2 + 3\operatorname{Ind}_{H_7}^G \theta_7^3\\
&\quad +4\operatorname{Ind}_{H_7}^G \theta_7^4 + 5\operatorname{Ind}_{H_7}^G \theta_7^5 + 6\operatorname{Ind}_{H_7}^G \theta_7^6).
\end{aligned}$$

Each piece is then computed from the induced characters in section 3.2. Γ_{H_2} depends on $q \pmod 8$.

For $q \equiv 1 \pmod 8$,

$$\Gamma_{H_2} = \frac{q-1}{4}\left[\frac{1}{2}(W' + W'') + \sum_{\beta} X_\beta + V + \sum_{\alpha(i)=1} W_\alpha\right] + \frac{q+3}{4}\sum_{\alpha(i)=-1} W_\alpha.$$

For $q \equiv 3 \pmod 8$,

$$\Gamma_{H_2} = \frac{q+1}{4}\left[\sum_{\beta(i)=1} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \frac{q-3}{4}\left[\frac{1}{2}(X' + X'') + \sum_{\beta(i)=-1} X_\beta\right].$$

For $q \equiv 5 \pmod 8$,

$$\Gamma_{H_2} = \frac{q-1}{4}\left[\sum_{\beta} X_\beta + V + \sum_{\alpha(i)=1} W_\alpha\right] + \frac{q+3}{4}\left[\frac{1}{2}(W' + W'') + \sum_{\alpha(i)=-1} W_\alpha\right].$$

And for $q \equiv 7 \pmod 8$,

$$\Gamma_{H_2} = \frac{q+1}{4}\left[\frac{1}{2}(X' + X'') + \sum_{\beta(i)=1} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \frac{q-3}{4}\sum_{\beta(i)=-1} X_\beta.$$

The contribution Γ_{H_3} of H_3 to the ramification module is

$$\Gamma_{H_3} = \frac{1}{3}\left(\operatorname{Ind}_{H_3}^G \theta_3 + 2\operatorname{Ind}_{H_3}^G \theta_3^2\right) = \operatorname{Ind}_{H_3}^G \theta_3,$$

since $\operatorname{Ind}_{H_3}^G \theta_3$ and $\operatorname{Ind}_{H_3}^G \theta_3^2$ are the same. This character was computed in section 3.2.

For H_7, the induced representations from the six nontrivial characters θ_7^k are not all the same. However, the representations $\mathrm{Ind}_{H_7}^G \theta_7^k$ and $\mathrm{Ind}_{H_7}^G \theta_7^{-k}$ are equal. Thus Γ_{H_7} is

$$\begin{aligned}
\Gamma_{H_7} &= \frac{1}{7}\left(\mathrm{Ind}_{H_7}^G \theta_7 + 2\,\mathrm{Ind}_{H_7}^G \theta_7^2 + \ldots + 6\,\mathrm{Ind}_{H_7}^G \theta_7^6\right) \\
&= \frac{1}{7}\left(7\,\mathrm{Ind}_{H_7}^G \theta_7 + 7\,\mathrm{Ind}_{H_7}^G \theta_7^2 + 7\,\mathrm{Ind}_{H_7}^G \theta_7^4\right) \\
&= \mathrm{Ind}_{H_7}^G \theta_7 + \mathrm{Ind}_{H_7}^G \theta_7^2 + \mathrm{Ind}_{H_7}^G \theta_7^4.
\end{aligned}$$

Recall from section 3.2 that the multiplicities of the irreducible representations W_α and X_β in the induced representation $\mathrm{Ind}_{H_7}^G \theta_7^k$ depend on the value of $\alpha(\phi)$ or $\beta(\phi)$, and that this value must be $e^{\frac{2\pi i k}{7}}$ for some $k = 0, \ldots, 6$. In the sum $\Gamma_{H_7} = \mathrm{Ind}_{H_7}^G \theta_7 + \mathrm{Ind}_{H_7}^G \theta_7^2 + \mathrm{Ind}_{H_7}^G \theta_7^4$ we will have, for example for the multiplicities of the W_α when $q \equiv 1 \pmod{28}$,

$$\begin{aligned}
\Gamma_{H_7} &= \mathrm{Ind}_{H_7}^G \theta_7 + \mathrm{Ind}_{H_7}^G \theta_7^2 + \mathrm{Ind}_{H_7}^G \theta_7^4 \\
&= \frac{q-1}{7} \sum_{\alpha(\phi)\neq e^{\pm\frac{2\pi i}{7}}} W_\alpha + \frac{q+6}{7} \sum_{\alpha(\phi)= e^{\pm\frac{2\pi i}{7}}} W_\alpha \\
&+ \frac{q-1}{7} \sum_{\alpha(\phi)\neq e^{\pm\frac{4\pi i}{7}}} W_\alpha + \frac{q+6}{7} \sum_{\alpha(\phi)= e^{\pm\frac{4\pi i}{7}}} W_\alpha \\
&+ \frac{q-1}{7} \sum_{\alpha(\phi)\neq e^{\pm\frac{8\pi i}{7}}} W_\alpha + \frac{q+6}{7} \sum_{\alpha(\phi)= e^{\pm\frac{8\pi i}{7}}} W_\alpha \\
&+ \text{ other characters.}
\end{aligned}$$

This adds up to

$$\Gamma_{H_7} = \frac{3q+4}{7} \sum_{\alpha(\phi)\neq 1} W_\alpha + \frac{3q-3}{7} \sum_{\alpha(\phi)=1} W_\alpha + \text{ other characters.}$$

The multiplicities of the other irreducible characters in $\mathrm{Ind}_{H_7}^G \theta_7^k$ do not depend on k. Adding these in, the total for the case $q \equiv 1 \pmod{28}$ is

$$\Gamma_{H_7} = \frac{3q-3}{7}\left[\sum_{\beta} X_\beta + V + \sum_{\alpha(\phi)=1} W_\alpha + \frac{1}{2}(W' + W'')\right] + \frac{3q+4}{7} \sum_{\alpha(\phi)\neq 1} W_\alpha.$$

Similar calculations yield the following. If $q \equiv 13 \pmod{28}$,

$$\Gamma_{H_7} = \frac{3q+3}{7}\left[\sum_{\beta(\phi)=1} X_\beta + V + \sum_{\alpha} W_\alpha + \frac{1}{2}(W' + W'')\right] + \frac{3q-4}{7} \sum_{\beta(\phi)\neq 1} X_\beta.$$

If $q \equiv 15 \pmod{28}$, we have

$$\Gamma_{H_7} = \frac{3q-3}{7}\left[\sum_{\beta} X_\beta + V + \sum_{\alpha(\phi)=1} W_\alpha + \frac{1}{2}(X' + X'')\right] + \frac{3q+4}{7}\sum_{\alpha(\phi)\neq 1} W_\alpha.$$

And if $q \equiv 27 \pmod{28}$, we have

$$\Gamma_{H_7} = \frac{3q+3}{7}\left[\sum_{\beta(\phi)=1} X_\beta + V + \sum_{\alpha} W_\alpha + \frac{1}{2}(X' + X'')\right] + \frac{3q-4}{7}\sum_{\beta(\phi)\neq 1} X_\beta.$$

To compute the ramification module, we sum the components Γ_{H_2}, Γ_{H_3}, and Γ_{H_7} listed above. The following numbers will be useful.

Definition 1. *For each possible equivalence class of $q \pmod{84}$, we define a **base multiplicity** m, as follows:*

- *If $q \equiv 1, 13, 29,$ or $43 \pmod{84}$, then $m = q + \lfloor \frac{q}{84} \rfloor$.*
- *If $q \equiv 41, 55, 71,$ or $83 \pmod{84}$, then $m = q + \lceil \frac{q}{84} \rceil$.*

Definition 2. *Let $\alpha : \mathbb{F}^\times \to \mathbb{C}^\times$ be a character of $\mathbb{F}^\times$. Then we define a number*

$$N_\alpha = \#\{x \in \{i, \omega, \phi\} \mid x \in \mathbb{F}^\times \text{ and } \alpha(x) \neq 1\}.$$

Definition 3. *Recall that T is the cyclic subgroup of $\mathbb{F}(\sqrt{\varepsilon})^\times$ of order $q+1$. Let $\beta : T \to \mathbb{C}^\times$ be a character of T. Then we define a number*

$$N_\beta = \#\{x \in \{i, \omega, \phi\} \mid x \in T \text{ and } \beta(x) \neq 1\}.$$

Theorem 4. *We have the following decomposition of the ramification module:*

- *If $q \equiv 1 \pmod{8}$, then*

$$\tilde{\Gamma}_G = \frac{m}{2}(W' + W'') + mV + \sum_{\beta}(m - N_\beta)X_\beta + \sum_{\alpha}(m + N_\alpha)W_\alpha$$

- *If $q \equiv 3 \pmod{8}$, then*

$$\tilde{\Gamma}_G = \frac{m-1}{2}(X' + X'') + mV + \sum_{\beta}(m - N_\beta)X_\beta + \sum_{\alpha}(m + N_\alpha)W_\alpha$$

- *If $q \equiv 5 \pmod{8}$, then*

$$\tilde{\Gamma}_G = \frac{m+1}{2}(W' + W'') + mV + \sum_{\beta}(m - N_\beta)X_\beta + \sum_{\alpha}(m + N_\alpha)W_\alpha$$

- *If $q \equiv 7 \pmod{8}$, then*

$$\tilde{\Gamma}_G = \frac{m}{2}(X' + X'') + mV + \sum_{\beta}(m - N_\beta)X_\beta + \sum_{\alpha}(m + N_\alpha)W_\alpha$$

4.2. **Equivariant degree.** Now we will define and compute the equivariant degree of a G-invariant divisor. (See for example [B] for more details). This, together with the equivariant Riemann-Roch formula (1), will allow us to compute the G-module structure of the Riemann-Roch space $L(D)$.

Fix a point $P \in X$ and let D be a divisor on X of the form

$$D = \frac{1}{e_P} \sum_{g \in G} g(P) = \sum_{g \in G/G_P} g(P),$$

where G_P denotes the stabilizer in G of P and $e_P = |G_P|$ denotes the ramification index at P. Such a divisor is called a *reduced orbit*; any G-invariant divisor on X can be written as a sum of multiples of reduced orbits.

The *equivariant degree* of a multiple rD of a reduced orbit is the virtual representation

$$\deg_{eq}(rD) = \begin{cases} \mathrm{Ind}_{G_P}^{G} \ \sum_{\ell=1}^{r} \theta_P^{-\ell}, & r > 0 \\ 0, & r = 0 \\ -\mathrm{Ind}_{G_P}^{G} \ \sum_{\ell=0}^{|r|-1} \theta_P^{-\ell}, & r < 0 \end{cases}$$

where θ_P is the ramification character of X at P (a nontrivial character of G_P). In general, the equivariant degree is additive on disjointly supported divisors. Note that if r is a multiple of e_P, then then D is the pull-back of a divisor on X/G via ψ in (3), and the equivariant degree is a multiple of the regular representation $\mathbb{C}[G]$ of G. More generally, if D is a reduced orbit and $r = e_P r' + r''$, then

$$\deg_{eq}(rD) = r' \cdot \mathbb{C}[G] + \deg_{eq}(r'' D).$$

(Note this is true even when r' is negative).

On the Hurwitz curve X, the results of section 2.2 tell us that there are only four types of reduced orbits to consider: the stabilizer G_P of a point P in the support of D may have order 1, 2, 3, or 7, and therefore be either trivial or conjugate to H_2, H_3, or H_7. Let D_1, D_2, D_3, and D_7 denote reduced orbits of each type. There is only one choice of reduced orbit for D_2, D_3, and D_7; for D_1 we see from the definition that the equivariant degree does not depend on our choice of orbit. Given a point in D_1, the stabilizer is trivial, so the divisor is a pullback and the equivariant degree is

$$\deg_{eq}(D_1) = \mathbb{C}[G].$$

A general G-invariant divisor may be written as $r_1 D_1 + r_2 D_2 + r_3 D_3 + r_7 D_7$. If we write $r_2 = 2r_2' + r_2''$, $r_3 = 3r_3' + r_3''$, and $r_7 = 7r_7' + r_7''$, then we have

$$\begin{aligned} &\deg_{eq}(r_1 D_1 + r_2 D_2 + r_3 D_3 + r_7 D_7) \\ &\quad = \deg_{eq}((r_1 + r_2' + r_3' + r_7')D_1 + r_2'' D_2 + r_3'' D_3 + r_7'' D_7) \\ &\quad = (r_1 + r_2' + r_3' + r_7')\mathbb{C}[G] + \deg_{eq}(r_2'' D_2 + r_3'' D_3 + r_7'' D_7). \end{aligned}$$

Therefore, to compute the equivariant degree of a general divisor, all that remains is to compute $\deg_{eq}(r_i D_i)$ for $i \in \{2, 3, 7\}$, where we may assume that $1 \le r_i < i$.

Case 1: : $r_2 D_2$. Given our assumptions, the only possibility is that $r_2 = 1$. Given a point P in the support of D_2, the stabilizer G_P is conjugate to H_2. In this case, the equivariant degree of D_2 is

$$\deg_{eq}(D_2) = \operatorname{Ind}_{H_2}^G \theta_2.$$

Case 2: : $r_3 D_3$. Here we may have either $r_3 = 1$ or $r_3 = 2$. The stabilizer of a point in the support of D_3 is conjugate to H_3. Recall that $\operatorname{Ind}_{H_3}^G \theta_3^2 = \operatorname{Ind}_{H_3}^G \theta_3$, so we have

$$\begin{aligned} \deg_{eq}(D_2) &= \operatorname{Ind}_{H_3}^G \theta_3 \\ \deg_{eq}(2D_2) &= 2 \operatorname{Ind}_{H_3}^G \theta_3. \end{aligned}$$

Case 3: : $r_7 D_7$. In this case, we have $1 \leq r_7 \leq 6$. The stabilizer of a point in the support of D_7 is conjugate to H_7. Recall that for $k = 1, \ldots, 6$, $\operatorname{Ind}_{H_7}^G \theta_7^k = \operatorname{Ind}_{H_7}^G \theta_7^{-k}$. Therefore the equivariant degree is as follows:

- $\deg_{eq}(D_7) = \operatorname{Ind}_{H_7}^G \theta_7$.
- $\deg_{eq}(2D_7) = \operatorname{Ind}_{H_7}^G \theta_7 + \operatorname{Ind}_{H_7}^G \theta_7^2$.
- $\deg_{eq}(3D_7) = \operatorname{Ind}_{H_7}^G \theta_7 + \operatorname{Ind}_{H_7}^G \theta_7^2 + \operatorname{Ind}_{H_7}^G \theta_7^3$, which is the same as the H_7 component of the ramification module, Γ_{H_7}.
- $\deg_{eq}(4D_7) = \Gamma_{H_7} + \operatorname{Ind}_{H_7}^G \theta_7^3$.
- $\deg_{eq}(5D_7) = \Gamma_{H_7} + \operatorname{Ind}_{H_7}^G \theta_7^3 + \operatorname{Ind}_{H_7}^G \theta_7^2$.
- $\deg_{eq}(6D_7) = 2\Gamma_{H_7}$.

Now we add these up. As in the case of the ramification module, the equivariant degree is most conveniently written in terms of a "base multiplicity" and modifiers. We define the base multiplicity as follows.

- If $q \equiv 1 \pmod 4$, then let $b_2 = r_2 \left(\frac{q-1}{2}\right)$. Otherwise, if $q \equiv 3 \pmod 4$, then let $b_2 = r_2 \left(\frac{q+1}{2}\right)$.
- If $q \equiv 1 \pmod 3$, then let $b_3 = r_3 \left(\frac{q-1}{3}\right)$, and if $q \equiv 2 \pmod 3$, then let $b_3 = r_3 \left(\frac{q+1}{3}\right)$.
- Similarly, if $q \equiv 1 \pmod 7$, then let $b_7 = r_7 \left(\frac{q-1}{7}\right)$, and if $q \equiv 6 \pmod 7$, then let $b_7 = r_7 \left(\frac{q+1}{7}\right)$.

The base multiplicity is then defined to be

$$\begin{aligned} b &= b_2 + b_3 + b_7 \\ &= r_2 \left(\frac{q \pm 1}{2}\right) + r_3 \left(\frac{q \pm 1}{3}\right) + r_7 \left(\frac{q \pm 1}{7}\right). \end{aligned}$$

Then the equivariant degree $\deg_{eq}(D)$ of the divisor $D = r_1 D_1 + r_2 D_2 + r_3 D_3 + r_7 D_7$, with $0 \leq r_2 \leq 1$, $0 \leq r_3 \leq 2$, and $0 \leq r_7 \leq 6$, is

$$\deg_{eq}(D) = b\left[\sum_{\beta} X_\beta + V + \sum_{\alpha} W_\alpha\right] + \text{ modifiers,} \tag{7}$$

where the modifiers are listed in the table below. For each q, three of the rows below will be added.

q	Modifiers to equivariant degree
$q \equiv 1 \pmod 8$	$+\, r_2 \sum_{\alpha(i)=-1} W_\alpha + \frac{b}{2}(W' + W'')$
$q \equiv 3 \pmod 8$	$-\, r_2 \sum_{\beta(i)=-1} X_\beta + \frac{b - r_2}{2}(X' + X'')$
$q \equiv 5 \pmod 8$	$+\, r_2 \sum_{\alpha(i)=-1} W_\alpha + \frac{b + r_2}{2}(W' + W'')$
$q \equiv 7 \pmod 8$	$-\, r_2 \sum_{\beta(i)=-1} X_\beta + \frac{b}{2}(X' + X'')$
$q \equiv 1 \pmod 3$	$+\, r_3 \sum_{\alpha(\omega)\neq 1} W_\alpha$
$q \equiv 2 \pmod 3$	$-r_3 \sum_{\beta(\omega)\neq 1} X_\beta$
$q \equiv 1 \pmod 7$	$+ \sum_{k=1}^{r_7} \sum_{\alpha(\phi)=e^{\pm\frac{2\pi i k}{7}}} W_\alpha$
$q \equiv 6 \pmod 7$	$- \sum_{k=1}^{r_7} \sum_{\beta(\phi)=e^{\pm\frac{2\pi i k}{7}}} X_\beta$

4.3. **The Riemann-Roch space.** Now we would like to compute the G-module structure of the Riemann-Roch space $L(D)$ for a G-invariant divisor D. First, let us consider which G-invariant divisors are non-special. To be non-special, it is sufficient to have $\deg D > 2g - 2$, where

$$g = 1 + \frac{(q)(q^2-1)}{168}$$

is the genus of X, so $2g - 2 = \frac{1}{84}q(q^2-1) = \frac{1}{168}|G|$. The reduced orbits D_1, D_2, D_3 and D_7 have degrees $|G|$, $|G|/2$, $|G|/3$, and $|G|/7$, respectively. Therefore if a G-invariant divisor $r_1D_1 + r_2D_2 + r_3D_3 + r_7D_7$ has positive degree, the smallest its degree could be is $|G|/42$, which is strictly larger than $2g-2$. Therefore any G-invariant divisor with positive degree is non-special.

Thus for any G-invariant divisor D with positive degree, we may use the equivariant Riemann-Roch formula (1) to compute the G-module structure of the Riemann-Roch space $L(D)$:

$$[L(D)] = (1 - g_{X/G})[\mathbb{C}[G]] + [\deg_{eq}(D)] - [\tilde{\Gamma}_G].$$

Since $X/G \cong \mathbb{P}^1$, its genus is zero. As in section 4.2, we may assume that $D = r_1D_1+r_2D_2+r_3D_3+r_7D_7$, with $0 \le r_2 \le 1$, $0 \le r_3 \le 2$, and $0 \le r_7 \le 6$. Combining the results and notation of sections 4.1 and 4.2, we obtain the following.

$$L(D) = (1+r_1)\mathbb{C}[G] + (b-m)\left[\sum_\beta X_\beta + V + \sum_\alpha W_\alpha\right] + \text{ modifiers},$$

where the modifiers depend on q (mod 168) and are listed in the following table. Again, for each value of q, three of the rows below will be added.

q	Modifiers to Riemann-Roch space
$q \equiv 1 \pmod 8$	$+ (r_2-1)\sum_{\alpha(i)=-1} W_\alpha + \frac{b-m}{2}(W'+W'')$
$q \equiv 3 \pmod 8$	$+ (1-r_2)\sum_{\beta(i)=-1} X_\beta + \frac{b-m+1-r_2}{2}(X'+X'')$
$q \equiv 5 \pmod 8$	$+ (r_2-1)\sum_{\alpha(i)=-1} W_\alpha + \frac{b-m+r_2-1}{2}(W'+W'')$
$q \equiv 7 \pmod 8$	$+ (1-r_2)\sum_{\beta(i)=-1} X_\beta + \frac{b-m}{2}(X'+X'')$
$q \equiv 1 \pmod 3$	$+ (r_3-1)\sum_{\alpha(\omega)\neq 1} W_\alpha$
$q \equiv 2 \pmod 3$	$+ (1-r_3)\sum_{\beta(\omega)\neq 1} X_\beta$
$q \equiv 1 \pmod 7$	$+\sum_{k=1}^{r_7}\sum_{\alpha(\phi)=e^{\pm\frac{2\pi ik}{7}}} W_\alpha - \sum_{alpha(\phi)\neq 1} W_\alpha$
$q \equiv 6 \pmod 7$	$+\sum_{\beta(\phi)\neq 1} X_\beta - \sum_{k=1}^{r_7}\sum_{\beta(\phi)=e^{\pm\frac{2\pi ik}{7}}} X_\beta$

4.4. **Action on holomorphic differentials.** As a corollary, it is an easy exercise now to compute explicitly the decomposition

$$H^1(X,\mathbb{C}) = H^0(X,\Omega^1) \oplus \overline{H^0(X,\Omega^1)} = L(K_X) \oplus \overline{L(K_X)},$$

into irreducible G-modules, where K_X is a canonical divisor of X. The action of G on the complex conjugate vector space $\overline{L(K_X)}$ of $L(K_X)$ will be by the complex conjugate (contragredient) representation. The Riemann-Hurwitz theorem tells us that

$$\begin{aligned} K_X &= \pi^*(K_{\mathbb{P}^1}) + R \\ &= -2D_1 + D_2 + 2D_3 + 6D_7 \end{aligned}$$

where R is the ramification divisor. Thus the equivariant degree of K_X is $\deg_{eq}(K_X) = -2\cdot\mathbb{C}[G] + \deg_{eq}(R)$. Note from the preliminary equivariant degree calculations, that

$$\begin{aligned}\deg_{eq}(R) &= \deg_{eq} D_2 + \deg_{eq} 2D_3 + \deg_{eq} 6D_7 \\ &= \mathrm{Ind}_{H_2}^G \theta_2 + 2\,\mathrm{Ind}_{H_3}^G \theta_3 + \sum_{k=1}^{6} \mathrm{Ind}_{H_7}^G \theta_7^k \\ &= 2\Gamma_{H_2} + 2\Gamma_{H_3} + 2\Gamma_{H_7} \\ &= 2\tilde{\Gamma}.\end{aligned}$$

Therefore, using the equivariant Riemann-Roch formula (1),

$$L(K_X) = \tilde{\Gamma} - \mathbb{C}[G]. \tag{8}$$

We will see in the next section that this is invariant under complex conjugation, so that as G-modules, $H^1(X, \mathbb{C}) \cong 2L(K_X)$.

Using the results of section 4.1, we obtain the following.

Theorem 5. *The G-module structure of $L(K) = H^0(X, \Omega^1)$ is as follows:*

- *If $q \equiv 1,\ 97,$ or $113 \pmod{168}$, then*

$$L(K_X) = \frac{\lfloor \frac{q}{84} \rfloor - 1}{2}(W' + W'') + \sum_\beta \left(\lfloor \frac{q}{84} \rfloor + 1 - N_\beta\right) X_\beta + \lfloor \frac{q}{84} \rfloor V + \sum_\alpha \left(\lfloor \frac{q}{84} \rfloor - 1 + N_\alpha\right) W_\alpha.$$

- *If $q \equiv 43 \pmod{168}$, then*

$$L(K_X) = \lfloor \frac{q}{84} \rfloor \left[\frac{1}{2}(X' + X'') + V\right] + \sum_\beta \left(\lfloor \frac{q}{84} \rfloor + 1 - N_\beta\right) X_\beta + \sum_\alpha \left(\lfloor \frac{q}{84} \rfloor - 1 + N_\alpha\right) W_\alpha.$$

- *If $q \equiv 13,\ 29,$ or $85 \pmod{168}$, then*

$$L(K_X) = \lfloor \frac{q}{84} \rfloor \left[\frac{1}{2}(W' + W'') + V\right] + \sum_\beta \left(\lfloor \frac{q}{84} \rfloor + 1 - N_\beta\right) X_\beta + \sum_\alpha \left(\lfloor \frac{q}{84} \rfloor - 1 + N_\alpha\right) W_\alpha.$$

- *If $q \equiv 127 \pmod{168}$, then*

$$L(K_X) = \frac{\lfloor \frac{q}{84} \rfloor + 1}{2}(X' + X'') + \sum_\beta \left(\lfloor \frac{q}{84} \rfloor + 1 - N_\beta\right) X_\beta + \lfloor \frac{q}{84} \rfloor V + \sum_\alpha \left(\lfloor \frac{q}{84} \rfloor - 1 + N_\alpha\right) W_\alpha.$$

- *If $q \equiv 41 \pmod{168}$, then*

$$L(K_X) = \frac{\lceil \frac{q}{84} \rceil - 1}{2}(W' + W'') + \sum_\beta \left(\lceil \frac{q}{84} \rceil + 1 - N_\beta\right) X_\beta + \lceil \frac{q}{84} \rceil V + \sum_\alpha \left(\lceil \frac{q}{84} \rceil - 1 + N_\alpha\right) W_\alpha.$$

- *If $q \equiv 83,\ 139,$ or $155 \pmod{168}$, then*

$$L(K_X) = \lceil \frac{q}{84} \rceil \left[\frac{1}{2}(X' + X'') + V\right] + \sum_\beta \left(\lceil \frac{q}{84} \rceil + 1 - N_\beta\right) X_\beta + \sum_\alpha \left(\lceil \frac{q}{84} \rceil - 1 + N_\alpha\right) W_\alpha.$$

- *If $q \equiv 125 \pmod{168}$, then*

$$L(K_X) = \lceil \frac{q}{84} \rceil \left[\frac{1}{2}(W' + W'') + V\right] + \sum_\beta \left(\lceil \frac{q}{84} \rceil + 1 - N_\beta\right) X_\beta + \sum_\alpha \left(\lceil \frac{q}{84} \rceil - 1 + N_\alpha\right) W_\alpha.$$

- *If* $q \equiv 55,\ 71,$ *or* $167 \pmod{168}$, *then*

$$L(K_X) = \frac{\lceil \frac{q}{84} \rceil + 1}{2}(X' + X'') + \sum_\beta \left(\lceil \frac{q}{84} \rceil + 1 - N_\beta \right) X_\beta + \lceil \frac{q}{84} \rceil V + \sum_\alpha \left(\lceil \frac{q}{84} \rceil - 1 + N_\alpha \right) W_\alpha.$$

5. Galois action

As discussed in section 3, there is a Galois action on the set of equivalence classes of irreducible representations of $PSL(2,q)$. One question of obvious interest is whether the modules we have computed are invariant under this action.

Theorem 6. *The ramification module is Galois-invariant.*

Proof. Recall from section 3 that the Galois group $\mathcal{G}$ permutes mth roots of unity, where $m = q(q^2-1)/4$. It acts on representations of $PSL(2,q)$ by permuting character values. Thus it fixes the trivial representation and the q-dimensional representation V, whose character values are rational. It will act as a permutation on the representations W_α and on the representations X_β. Lastly, it will act as an involution on either the representations W' and W'' or X' and X''.

Because the multiplicities of W' and W'' or X' and X'' are the same in the ramification module, the Galois action will be invariant on this component. The multiplicity of a representation W_α or X_β in the ramification module depends on the number N_α or N_β, which is determined by the value of the character α or β on the special numbers i, ω, and ϕ. In fact, the numbers N_α and N_β are determined only by whether these character values are equal to 1 or not equal to 1. Since an element of the Galois group will take a character value to a power of itself, the Galois action must preserve the numbers N_α and N_β. Therefore this component of the ramification module is invariant as well.

□

Since the ramification module is Galois-invariant, and of course the regular representation is Galois-invariant, $L(K_X)$ will be Galois invariant. In particular, as stated in section 4.4, $L(K_X)$ will be invariant under complex conjugation. For a general divisor D, the Riemann-Roch space $L(D)$ will be Galois-invariant if and only if the equivariant degree of D is.

Theorem 7. *Let* $D = r_1D_1 + r_2D_2 + r_3D_3 + r_7D_7$ *be a* G*-invariant divisor. Then the equivariant degree of* D *is Galois-invariant if* $r_7 \in \{0,3,6\} \pmod 7$.

Proof. As in section 4.2, multiples of 2 in r_2, 3 in r_3, and 7 in r_7 can be absorbed into the r_1D_1 term without affecting the equivariant degree. Therefore we may assume that $0 \le r_2 \le 1$, $0 \le r_3 \le 2$, and $0 \le r_7 \le 6$.

The result can again be seen by looking at the multiplicities of representations permuted by the Galois group. The multiplicities of W' and W'' or X' and X'' are the same. By (7), the multiplicity of a representation W_α or X_β depends on r_2, r_3, and r_7, and not on r_1. Again, the Galois action will not permute a representation W_α with $\alpha(i) = 1$ with one with $\alpha(i) \neq 1$; similarly for X_β, and for ω. However, it could permute for example a representation W_α with $\alpha(\phi) = e^{\frac{2\pi i}{7}}$ with one with $\alpha(\phi) = e^{\frac{4\pi i}{7}}$. Thus the equivariant degree may not be Galois-invariant unless the multiplicities of these representations are equal. In the cases where $r_7 \in \{0,3,6\}$, then these multiplicities will be equal; otherwise they will not.

□

Note that for some values of q, the equivariant degree may be Galois-invariant even if r_7 is not 0, 3, or 6.

A previous result of the first two authors (see [JK1]) gives a simpler formula (see equation 2) to compute the multiplicity of an irreducible representation in the ramification module, when the ramification module is Galois-invariant. In the example at hand, if $r_7 \in \{0, 3, 6\}$, then since the equivariant degree is a multiple of the H_7 component of the ramification module, a slight modification of this formula gives an easy computation of the equivariant degree and therefore the Riemann-Roch space.

Corollary 8. *Let $D = r_1D_1 + r_2D_2 + r_3D_3 + r_7D_7$, with $0 \le r_2 \le 1$, $0 \le r_3 \le 2$, and $r_7 \in \{0, 3, 6\}$. Then*

$$\begin{aligned} L(D) &= \bigoplus_{\pi \in G^*} \Big[(1 + r_1 + r_2 + \frac{r_3}{2} + \frac{r_7}{6}) \dim \pi \\ &\quad + (\frac{1}{2} - r_2) \dim \pi^{H_2} + (\frac{1}{2} - \frac{r_3}{2}) \dim \pi^{H_3} + (\frac{1}{2} - \frac{r_7}{6}) \dim \pi^{H_7}\Big] \pi. \end{aligned}$$

Note that in spite of appearances, the multiplicity of each irreducible representation will in fact be an integer.

Proof. We see from the calculations in section 4.2 that the equivariant degree of D is equal to

$$\begin{aligned} \deg_{eq}(D) &= r_1\mathbb{C}[G] + 2r_2\Gamma_{H_2} + r_3\Gamma_{H_3} + \frac{r_7}{3}\Gamma_{H_7} \\ &= \bigoplus_{\pi \in G^*} \Big[(r_1 + r_2 + \frac{r_3}{2} + \frac{r_7}{6}) \dim \pi \\ &\quad - r_2 \dim \pi^{H_2} - \frac{r_3}{2} \dim \pi^{H_3} - \frac{r_7}{6} \dim \pi^{H_7}\Big] \pi. \end{aligned}$$

The ramification module is

$$\tilde{\Gamma}_G = \bigoplus_{\pi \in G^*} \left[\sum_{\ell \in 2,3,7} (\dim \pi - \dim (\pi^{H_\ell}))\frac{1}{2} \right] \pi.$$

This sum splits into $\tilde{\Gamma}_G = \Gamma_{H_2} + \Gamma_{H_3} + \Gamma_{H_7}$ in the obvious way along the inner sum. Putting these together using the equivariant Riemann-Roch formula (1), we obtain the desired result.

□

References

[B] N. Borne, "Une formule de Riemann-Roch equivariante pour des courbes," Can. J. Math. **55**(2003)693-710. (see also thesis, Univ. Bordeaux, 1999. Available on the web at `http://www.dm.unibo.it/~borne/`)

[C] M. Conder, "Hurwitz groups: a brief survey," Bull. Amer. Math. Soc. 23(2) (1990) 359–370.

[EL] G. Ellingsrud and K. Lønsted, "An equivariant Lefschetz trace formula for finite reductive groups," Math Ann **251**(1980), 253-261.

[FH] W. Fulton and J. Harris, **Representation theory: a first course**, Springer-Verlag, 1991.

[GAP] The GAP Group, **GAP – Groups, Algorithms, and Programming, Version 4.4**; 2002, (`http://www.gap-system.org`).

[Ja] G. Janusz, "Simple components of $\mathbb{Q}[SL(2, q)]$," Comm. Alg. 1(1974)1-22.

[JK1] D. Joyner and A. Ksir, "Decomposing representations of finite groups on Riemann-Roch spaces," to appear in *Proceedings of the American Mathematical Society*, math.AG/0312383.

[JK2] ——, "Modular representations on some Riemann-Roch spaces of modular curves $X(N)$," in **Computational Aspects of Algebraic Curves**, (Editor: T. Shaska) Lecture Notes in Computing, WorldScientific, 2005, math.AG/0502586.

[K] E. Kani, "The Galois-module structure of the space of holomorphic differentials of a curve," J. Reine Angew. Math. **367** (1986), 187-206.

[L] M. Larsen, "How often is $84(g-1)$ achieved?" Israel J. Math. 126 (2001) 1–16.

[M] A.M. Macbeath, "On a theorem of Hurwitz," Proc. Glasgow Math. Assoc. 5 (1961) 90–96.

[N] S. Nakajima, "Galois module structure of cohomology groups for tamely ramified coverings of algebraic varieties," J. Number Theory 22 (1986) 115-123.

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