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MAPPINGS AND PAIRWISE CONTINUITY ON PAIRWISE LINDELÖF BITOPOLOGICAL SPACES

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Abstract. In this paper, we shall continue the study of pairwise Lindelöf bitopological spaces initiated by Fora and Hdeib. Furthermore, we introduce the concepts of pairwise continuity, pairwise open and pairwise homeomorphism, and the properties for them are established. We also shows that a Lindelöf space and a B-Lindelöf space are both bitopological property and p-topological property.

1. Introduction

A bitopological space (X, τ_1, τ_2) is a set X together with two topologies τ_1 and τ_2 defined on X. The reader is suggested to refer to [3] for the detail definitions, terminology and notation.

Nowadays mapping and continuity stand among the most important topics and most researched points in topology. It is being studied by many general topologists including the authors. In this paper we extend the idea of continuity in a topological space to a bitopological space. We also extend the result for the continuity in a topological space to the pairwise continuity in a bitopological space and study their properties.

The purpose of this paper is to study the effect of mappings and pairwise continuity on pairwise Lindelöf bitopological spaces. These spaces were introduced by Fora and Hdeib [2]. We show that some mappings preserve these properties (i.e., Lindelöf property and B-Lindelöf property). The main results in our study are that the image of a Lindelöf bitopological space under a continuous or a p-continuous function is Lindelöf, and the image of a B-Lindelöf space under a continuous and open function or a p-continuous and p-open function is B-Lindelöf.

2. Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) always mean topological spaces and bitopological spaces, respectively. In this paper, we shall use p- to denote pairwise. For instance, p-continuous stands for pairwise continuous. We always use (τ_i, τ_j)- to denote certain properties with respect to topology τ_i and τ_j in bitopological spaces, where $i, j \in \{1, 2\}$. By τ_i-open set, we shall mean the open set with respect to topology τ_i in X. By τ_i-open cover of X, we mean that the cover of X by τ_i-open sets in X. Sometimes the prefixes (τ_i, τ_j)- or τ_i- will

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be replaced by (i,j)- or i-, respectively, if there is no chance for confusion. The reader may consult [1] for the detail notations. The authors sometime write the term "pairwise Lindelöf spaces" meaning pairwise Lindelöf bitopological spaces.

Definition 1 (see [2, 4])**.** *A bitopological space (X,τ_1,τ_2) is said to be Lindelöf if the topological space (X,τ_1) and (X,τ_2) are both Lindelöf. Equivalently, (X,τ_1,τ_2) is i-Lindelöf if the topological space (X,τ_i) is Lindelöf. X is said Lindelöf if it is i-Lindelöf for each $i=1,2$, or, if every i-open cover of X has a countable subcover for each $i=1,2$.*

Definition 2 (see [2])**.** *A bitopological space (X,τ_1,τ_2) is called (i,j)-Lindelöf if for every i-open cover of X there is a countable j-open subcover. X is called B-Lindelöf if it is both (i,j)-Lindelöf and (j,i)-Lindelöf.*

Kopperman [5] and Tallafha et. al. [6], was mentioned about pairwise continuous functions and pairwise open functions on bitopological spaces. Tallafha et. al. [6] has given more definitions, i.e., pairwise closed and pairwise homeomorphism functions. The following assert the definition of continuous function in the sense of Tallafha et. al.

Definition 3. *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be two bitopological spaces. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be continuous if the functions $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ are both continuous. Equivalently, a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called i-continuous if the function $f:(X,\tau_i)\to(Y,\sigma_i)$ is continuous. f is said continuous if it is i-continuous for each $i=1,2$.*

Next we are going to define the second concept of pairwise continuous function on bitopological spaces in the sense of Tallafha et. al.

Definition 4. *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be two bitopological spaces. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called (i,j)-continuous if the function $f:(X,\tau_i)\to(Y,\sigma_j)$ is continuous. The function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called p-continuous if it is both (i,j)-continuous and (j,i)-continuous. Equivalently, a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is p-continuous if the functions $f:(X,\tau_1)\to(Y,\sigma_2)$ and $f:(X,\tau_2)\to(Y,\sigma_1)$ are both continuous.*

The following definitions are given two concepts of pairwise open and pairwise closed functions in the sense of Tallafha et. al.

Definition 5. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be open (resp. closed) if the functions $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ are both open (resp. closed). Equivalently, a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called i-open (resp. i-closed) if the function $f:(X,\tau_i)\to(Y,\sigma_i)$ is open (resp. closed). f is said open (resp. closed) if f is i-open (resp. i-closed) for each $i=1,2$.*

Definition 6. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called (i,j)-open (resp. (i,j)-closed) if the function $f:(X,\tau_i)\to(Y,\sigma_j)$ is open (resp. closed). f is said p-open (resp. p-closed) if it is both (i,j)-open (resp. (i,j)-closed) and (j,i)-open (resp. (j,i)-closed). Equivalently, a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be p-open (resp. p-closed) if the functions $f:(X,\tau_1)\to(Y,\sigma_2)$ and $f:(X,\tau_2)\to(Y,\sigma_1)$ are both open (resp. closed).*

Example 1. *Consider $X=\{a,b,c,d\}$ with τ_1 the discrete topology and topology $\tau_2=\{\emptyset,\{a\},\{a,b\},\{a,b,c\},X\}$ on X, and $Y=\{x,y,z,w\}$ with topologies $\sigma_1=$*

$\{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z,w\}, Y\}$ and $\sigma_2 = \{\emptyset, \{x\}, \{y,z,w\}, Y\}$ on Y. Define a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = y, f(b) = f(d) = z$ and $f(c) = w$. Observe that the functions $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ are continuous. Therefore the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is continuous. On the other hand, the functions $f : (X, \tau_1) \to (Y, \sigma_2)$ and $f : (X, \tau_2) \to (Y, \sigma_1)$ are also continuous. Therefore the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is p-continuous.

Example 2. *Consider (X, τ_1, τ_2) and (Y, σ_1, σ_2) are bitopological spaces as in Example 1. Define a function $g : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $g(a) = g(b) = x, g(c) = z$ and $g(d) = w$. The function $g : (X, \tau_1) \to (Y, \sigma_1)$ is continuous and $g : (X, \tau_2) \to (Y, \sigma_2)$ is not continuous since $\{y,z,w\} \in \sigma_2$ but its inverse image $g^{-1}(\{y,z,w\}) = \{c,d\} \notin \tau_2$. Thus $g : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not continuous. On the other hand, the function $g : (X, \tau_1) \to (Y, \sigma_2)$ is continuous and $g : (X, \tau_2) \to (Y, \sigma_1)$ is not continuous since $\{y,z,w\} \in \sigma_1$ but $g^{-1}(\{y,z,w\}) = \{c,d\} \notin \tau_2$. Thus $g : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not p-continuous.*

Example 3. *Consider a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ as in Example 1. Observe that the function $f : (X, \tau_2) \to (Y, \sigma_2)$ is not open since $\{a\} \in \tau_2$ but $f(\{a\}) = \{y\} \notin \sigma_2$. Thus $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not open. The function $f : (X, \tau_2) \to (Y, \sigma_1)$ is not open since $\{a,b\} \in \tau_2$ but $f(\{a,b\}) = \{y,z\} \notin \sigma_1$. Thus $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not p-open.*

3. Some Results on Pairwise Continuity

In this section, we are going to study the images for each type of pairwise Lindelöf spaces under several types of combinations of pairwise continuous and pairwise open functions. We shows that some mappings preserve certain type of pairwise Lindelöf spaces and the other, the images are another type of pairwise Lindelöf spaces.

Theorem 1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an i-continuous (resp. (j,i)-continuous) and surjective function. If (X, τ_1, τ_2) is τ_i-Lindelöf (resp. τ_j-Lindelöf), then (Y, σ_1, σ_2) is σ_i-Lindelöf.*

Proof. Let $\{G_k : k \in \Delta\}$ be a σ_i-open cover of Y, i.e., $Y = \bigcup_{k\in\Delta} G_k$ where $G_k \in \sigma_i$. Since $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-continuous (resp. (j,i)-continuous), then $f^{-1}(G_k) \in \tau_i$ (resp. $f^{-1}(G_k) \in \tau_j$) and $X = f^{-1}(Y) = \bigcup_{k\in\Delta} f^{-1}(G_k)$. Hence $\left\{f^{-1}(G_k) : k \in \Delta\right\}$ is a τ_i-open (resp. τ_j-open) cover of X. Since (X, τ_1, τ_2) is τ_i-Lindelöf (resp. τ_j-Lindelöf), so there exists a countable τ_i-open (resp. τ_j-open) subcover of X, say $\left\{f^{-1}(G_{k_n}) : n \in \mathbb{N}\right\}$ such that $X = \bigcup_{n\in\mathbb{N}} f^{-1}G_{k_n}$. Since f is surjective, $Y = f(X) = \bigcup_{n\in\mathbb{N}} f\left(f^{-1}(G_{k_n})\right) \subseteq \bigcup_{n\in\mathbb{N}} G_{k_n}$. Thus we obtain $\{G_{k_n} : n \in \mathbb{N}\}$ is a countable σ_i-open subcover of Y. Therefore (Y, σ_1, σ_2) is σ_i-Lindelöf. □

Corollary 1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a continuous (resp. p-continuous) and surjective function. If (X, τ_1, τ_2) is Lindelöf, then (Y, σ_1, σ_2) is Lindelöf.*

Theorem 2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an i-continuous (resp. (i,j)-continuous), surjective and j-open (resp. (j,i)-open) function. If (X, τ_1, τ_2) is (τ_i, τ_j)-Lindelöf, then (Y, σ_1, σ_2) is (σ_i, σ_j)-Lindelöf (resp. (σ_j, σ_i)-Lindelöf).*

Proof. Let $\{G_k : k \in \Delta\}$ is a σ_i-open (resp. σ_j-open) cover of Y. Following the proof of Theorem 1, $\left\{f^{-1}(G_k) : k \in \Delta\right\}$ is a τ_i-open cover of X. Since (X, τ_1, τ_2) is (τ_i, τ_j)-Lindelöf, so the τ_i-open cover of X has a countable τ_j-open subcover, say $\left\{f^{-1}(G_{k_n}) : n \in \mathbb{N}\right\}$ such that $X = \bigcup_{n \in \mathbb{N}} f^{-1}(G_{k_n})$. Since f is surjective, $Y = f(X) = \bigcup_{n \in \mathbb{N}} f\left(f^{-1}(G_{k_n})\right) \subseteq \bigcup_{n \in \mathbb{N}} G_{k_n}$. Thus we obtain $\{G_{k_n} : n \in \mathbb{N}\}$ is a countable subfamily by σ_j-open (resp. σ_i-open) sets which cover Y since f is a j-open (resp. (j,i)-open) function. Therefore (Y, σ_1, σ_2) is (σ_i, σ_j)-Lindelöf (resp. (σ_j, σ_i)-Lindelöf). □

Corollary 2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a continuous (resp. p-continuous), surjective and open (resp. p-open) function. If (X, τ_1, τ_2) is B-Lindelöf, then (Y, σ_1, σ_2) is B-Lindelöf.*

Theorem 3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an i-continuous (resp. (j,i)-continuous), surjective and (i,j)-open (resp. j-open) function. If (X, τ_1, τ_2) is τ_i-Lindelöf (resp. τ_j-Lindelöf), then (Y, σ_1, σ_2) is (σ_i, σ_j)-Lindelöf.*

Proof. Let $\{G_k : k \in \Delta\}$ be a σ_i-open cover of Y. Since $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-continuous (resp. (j,i)-continuous), then $\left\{f^{-1}(G_k) : k \in \Delta\right\}$ is a τ_i-open (resp. τ_j-open) cover of X. Since (X, τ_1, τ_2) is τ_i-Lindelöf (resp. τ_j-Lindelöf), so there exists a countable τ_i-open (resp. τ_j-open) subcover of X, say $\left\{f^{-1}(G_{k_n}) : n \in \mathbb{N}\right\}$. Since f is surjective and (i,j)-open (resp. j-open) function, we obtain $\{G_{k_n} : n \in \mathbb{N}\}$ is a countable subfamily by σ_j-open sets which also cover Y. This shows that (Y, σ_1, σ_2) is (σ_i, σ_j)-Lindelöf. □

Corollary 3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a continuous (resp. p-continuous), surjective and p-open (resp. open) function. If (X, τ_1, τ_2) is Lindelöf, then (Y, σ_1, σ_2) is B-Lindelöf.*

Theorem 4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an i-continuous (resp. (j,i)-continuous) and surjective function. If (X, τ_1, τ_2) is (τ_i, τ_j)-Lindelöf (resp. (τ_j, τ_i)-Lindelöf), then (Y, σ_1, σ_2) is σ_i-Lindelöf.*

Proof. Let $\{G_k : k \in \Delta\}$ be a σ_i-open cover of Y. Since $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-continuous (resp. (j,i)-continuous), then $\left\{f^{-1}(G_k) : k \in \Delta\right\}$ is a τ_i-open (resp. τ_j-open) cover of X. Since (X, τ_1, τ_2) is (τ_i, τ_j)-Lindelöf (resp. (τ_j, τ_i)-Lindelöf), so the τ_i-open (resp. τ_j-open) cover of X has a countable τ_j-open (resp. τ_i-open) subcover, say $\left\{f^{-1}(G_{k_n}) : n \in \mathbb{N}\right\}$. Since f is surjective, we obtain $\{G_{k_n} : n \in \mathbb{N}\}$ is a countable σ_i-open subcover of Y. This shows that (Y, σ_1, σ_2) is σ_i-Lindelöf. □

Corollary 4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a continuous (resp. p-continuous) and surjective function. If (X, τ_1, τ_2) is B-Lindelöf, then (Y, σ_1, σ_2) is Lindelöf.*

The concept of homeomorphism is well known in topological spaces. Now we extend this concept to bitopological spaces in the following definitions in sense of Tallafha et. al. [6].

Definition 7. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called i-homeomorphism if the function $f : (X, \tau_i) \to (Y, \sigma_i)$ is homeomorphism, or equivalently, if $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*

is bijection, i-continuous and $f^{-1} : (Y, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is i-continuous. The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are then called i-homeomorphic.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called homeomorphism if the function $f : (X, \tau_i) \to (Y, \sigma_i)$ is homeomorphism for each $i = 1, 2$, or equivalently, if $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is bijection, continuous and $f^{-1} : (Y, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is continuous. The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are then called homeomorphic.

In the following definition is given the second type of pairwise homeomorphism functions in the sense of Tallafha et. al. [6].

Definition 8. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (i, j)-homeomorphism if the functions $f : (X, \tau_i) \to (Y, \sigma_j)$ is homeomorphism, or equivalently, if $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is bijection, (i, j)-continuous and $f^{-1} : (Y, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is (i, j)-continuous. The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are then called (i, j)-homeomorphic.*

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called p-homeomorphism if the function f is both (i, j)-homeomorphism and (j, i)-homeomorphism, or equivalently, if $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is bijection, p-continuous and $f^{-1} : (Y, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is p-continuous. The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are then called p-homeomorphic.

Tallafha et. al. [6] use notation p_1- instead of p- for our definition of second type of pairwise. While Fora and Hdeib [2], and Kiliçman and Salleh [4] use notation p- instead of our definition of first type of pairwise. For instance Fora and Hdeib use p-continuous instead of continuous.

The concept of p- that was defined throughout this paper depends heavily on the order of the topologies, that is, (X, τ_1, τ_2) is different from (X, τ_2, τ_1). For instance, $(\mathbb{R}, \tau_u, \tau_{cof})$ is p-homeomorphic to $(\mathbb{R}, \tau_{cof}, \tau_u)$ but it is not p-homeomorphic to itself, where τ_u and τ_{cof} are usual topology and cofinite topology on $\mathbb{R}$, respectively. However, a bitopological space (X, τ_1, τ_2) is p-homeomorphic to itself if and only if (X, τ_1) is homeomorphic to (X, τ_2). Similar for the p-open, p-closed and p-continuous functions.

Example 4. *The function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ in Example 1 is not homeomorphism and not p-homeomorphism since $f^{-1} : (Y, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is not continuous and not p-continuous.*

Recall that, a property $\mathcal{P}$ of sets is called topological property if whenever a topological space (X, τ) has property $\mathcal{P}$, then every space homeomorphic to (X, τ) also has property $\mathcal{P}$. In the case of bitopological space (X, τ_1, τ_2), there are two types of topological properties since we have two types of homeomorphism. A property $\mathcal{P}$ will be called i-topological property (resp. (i, j)-topological property) if whenever (X, τ_1, τ_2) has property $\mathcal{P}$, then every space i-homeomorphic (resp. (i, j)-homeomorphic) to (X, τ_1, τ_2) also has property $\mathcal{P}$. If homeomorphism (resp. p-homeomorphism) considered (for the pairwise topology), we will call such property $\mathcal{P}$ as bitopological property (resp. p-topological property).

Utilizing Theorem 1, Corollary 1 and Corollary 2 we easily obtain the following corollary.

Corollary 5. *An i-Lindelöf property is i-topological property, a Lindelöf property and a B-Lindelöf property are both bitopological property and p-topological property.*

One question may be asked concerning the idea of n-topological space. What happens if we consider $n > 2$? This n-topological space can be constructed with the same idea, and the question in the 2-topological setup do extend naturally. A more abstract and may be less relevant generalization would be a N-topological space when X has continuum elements and N means a countable number of topologies residing on X. A difference may be arise for the last consideration.

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