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ON GENERALIZED QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce and study a class $\tilde{Q}_k(\alpha, \beta, \rho, \gamma)$ of analytic functions in the unit disc. This class generalizes the concept of quasi-convexity. Inclusion results, distortion theorem and some other properties of this class are investigated.

1. Introduction

Let $\tilde{P}(\gamma)$ denote the class of functions p of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots, \tag{1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and which satisfy the condition $|\arg p(z)| \leq \frac{\pi\gamma}{2}$ for some $\gamma(\gamma > 0)$ in E. We note that $\tilde{P}(1) \equiv P$ is the class of analytic functions with positive real part. It can easily be shown that the class $\tilde{P}(\gamma)$ is a convex set.

Let $V_k(\rho), k \geq 2, 0 \leq \rho < 1$, be the class of functions of analytic and locally univalent in E, $\quad f(0) = 0, \quad f'(0) = 1$ and satisfying the condition

$$\int_0^{2\pi} \left| \{Re \frac{(zf'(z))'}{f'(z)} - \rho\}/(1-\rho) \right| d\theta \leq k\pi. \tag{2}$$

When $\rho = 0$, we obtain the class $V_k, \quad (k \geq 2)$ of functions of bounded boundary rotation. It can easily be shown that $f \in V_k(\rho)$ if and only if there exists a function $f_1 \in V_k$ such that

$$f'(z) = (f_1'(z))^{1-\rho}. \tag{3}$$

We note that $V_2 \equiv C \subset S^\star$, where C and $S^\star$ are respectively the classes of convex and starlike univalent functions in E.

We now introduce the following classes of analytic functions.

Definition 1.1. Let $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then, for $0 \leq \rho < 1, \quad 0 \leq \gamma \leq 1, \quad f \in T_k^\star(\rho, \gamma)$ if and only if there exists a function $g \in V_k(\rho)$ such that, for $z \in E$, $\frac{f'(z)}{g'(z)} \in \tilde{P}(\gamma)$.

We note that $T_2^\star(\rho, \gamma) = \tilde{K}(\rho, \gamma) \subset \tilde{K}(\gamma)$, where $\tilde{K}(\gamma)$ is the class of strongly close-to-convex functions.

Definition 1.2. Let $\alpha, \beta \geq 0, \quad (\alpha + \beta \neq 0)$, and let f be analytic in E with $f(0) = 0, \quad f'(0) = 1$. Then $\quad f \in \tilde{Q}_k(\alpha, \beta, \rho, \gamma)$ for $z \in E$, if and only if there

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exists a $g \in V_k(\rho)$ such that

$$(4) \qquad \left\{\frac{\alpha}{\alpha+\beta}\frac{f'(z)}{g'(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{g'(z)}\right\} \in \tilde{P}(\rho), \quad \text{for some} \quad \gamma > 0.$$

The class $\tilde{Q_2}(\alpha, 0, 0, \gamma)$ consists of strongly close-to-convex functions. Also $\tilde{Q_2}(0, 1, 0, 1) \equiv C^\star$ is the class of quasi-convex functions introduced in [1]. Also see [2,3]. For $\beta = (1-\alpha)$, $g \in V_2(0) \equiv C$, we obtain the class of strongly α-quasi-convex functions discussed in [7]. The case $\rho = \beta = 0$, $\alpha = \gamma = 1$ gives us the class T_k which was introduced and investigated in [4]. We also refer to [5] for more details.

2. Main Results

Theorem 2.1. *Let f be analytic in E with $f(0) = f'(0) - 1 = 0$. Then $f \in \tilde{Q_k}(\alpha, \beta, \rho, \gamma)$ if and only if*

$$\left\{\frac{\alpha}{\alpha+\beta}f(z) + \frac{\beta}{\alpha+\beta}zf'(z)\right\} \in T_k^\star(\rho, \gamma), \quad \textit{for} \quad z \in E.$$

Proof. The proof follows immediately from the definition of these classes. □

Theorem 2.2. *For $\beta > 0$, $f \in \tilde{Q_k}(\alpha, \beta, \rho, \gamma)$ if and only if there exists $F \in T_k^\star(\rho, \gamma)$ such that*

$$(5) \qquad f(z) = \frac{\alpha+\beta}{\beta} z^{\frac{-\alpha}{\beta}} \int_0^z t^{\frac{\alpha}{\beta}-1} F(t)dt.$$

Proof. From (2.1), we have

$$F(z) = \frac{\alpha}{\alpha+\beta}f(z) + \frac{\beta}{\alpha+\beta}zf'(z),$$

and, using Theorem 2.1, we prove the result. □

Theorem 2.3. *Let $f \in \tilde{Q_k}(\alpha, \beta, \rho, \gamma)$, $\alpha, \beta > 0$. Then, for $|z| = r$ $(0 < r < 1)$, we have*

$$|f(z)| \geq \frac{\alpha+\beta}{2A}\left[\frac{1}{\alpha} - \frac{r^{\frac{-\alpha}{\beta}}}{\beta}G(\left(\frac{\alpha}{\beta}, A, B, -r\right)\right],$$

where

$$(6) \qquad A = \left(\frac{k}{2} - 1\right)(1-\rho) + \gamma + 1, \quad B = A + \frac{\alpha}{\beta},$$

G denotes the hypergeometric function and it is known to be analytic in E. This result is sharp as can be seen from the function $f_0 \in \tilde{Q_k}(\alpha, \beta, \rho, \gamma), \alpha, \beta >$ defined by

$$(7) \qquad f_0(z) = \frac{(\alpha+\beta)}{\beta(k+2\gamma)} z^{\frac{-\alpha}{\beta}} \int_0^z \xi^{\frac{\alpha}{\beta}-1}\left\{1 - \left(\frac{1-\xi}{1+\xi}\right)^{\frac{k}{2}+\gamma}\right\} d\xi.$$

Proof. We consider the straight line Γ joining 0 to $f(z) = Re^{i\phi}$. Γ is the image of a Jordan arc Γ in E connecting 0 to $z = re^{i\theta}$. The image of Γ under the mapping $\left|z^{\frac{\alpha}{\beta}}f(z)\right|$ will consist of many line-segments emanating from the origin each of length

$$r^{\frac{\alpha}{\beta}}R = \left|z^{\frac{\alpha}{\beta}}f(z)\right| = \int_\Gamma \left|\frac{d}{d\xi}[\xi^{\frac{\alpha+\beta}{\beta}}f(\xi)]\right| |d\xi|.$$

Since f is in $\tilde{Q}_k(\alpha,\beta,\rho,\gamma)$, there exists $F \in T_k^\star(\rho,\gamma)$ such that

$$\frac{d}{d\xi}\left[\xi^{\frac{\alpha}{\beta}} f(\xi)\right] = \frac{1}{\beta} f^{\frac{\alpha}{\beta}-1} F(\xi).$$

Thus, if $t = |\xi|$, we deduce that

$$r^{\frac{\alpha}{\beta}} R = \frac{\alpha+\beta}{\beta}\int_\Gamma \left|\xi^{\frac{\alpha}{\beta}-1} F(\xi)\right| |d\xi|. \tag{8}$$

Now, for $F \in T_k^\star(\rho,\gamma)$, we have

$$|F(z)| \geq \frac{1}{2A}\left[1-\left(\frac{1-r}{1+r}\right)^A\right], \tag{9}$$

where A is defined by (2.2) and we have used (1.3) together with a result proved in [7]. Using (2.5) in (2.4), we have

$$\begin{aligned} R = |f(z)| \quad &\geq \quad \frac{r^{\frac{-\alpha}{\beta}}(\alpha+\beta)}{-2\beta A}\int_0^r t^{\frac{\alpha}{\beta}-1}\left[1-\left(\frac{1-t}{1+t}\right)^A\right]dt \\ &= \quad \frac{(\alpha+\beta)}{2A}\left[\frac{1}{\alpha}-\frac{1}{\beta} r^{\frac{-\alpha}{\beta}}\int_0^r t^{\frac{\alpha}{\beta}-1}(1-t)^A(1+t)^{-A}dt\right] \\ &= \quad \frac{(\alpha+\beta)}{2A}\left[\frac{1}{\alpha}-\frac{r^{\frac{-\alpha}{\beta}}}{\beta}.G(\frac{\alpha}{\beta},A,B,-r)\right]. \end{aligned}$$

This completes the proof. □

Letting $r \longrightarrow 1$ in Theorem 2.3, we obtain the following result.

Theorem 2.4. *Let $f \in \tilde{Q}_k(\alpha,\beta,\rho,\gamma)$, $(\alpha,\beta > 0)$. Then $f(E)$ contains the schliht disc*

$$|z| < \frac{\alpha+\beta}{(k-2)(1-\rho)+2\gamma+2}.$$

We now have the following.

Theorem 2.5. *A function $f \in \tilde{Q}_k(\alpha,\beta,0,\gamma)$ for $\alpha,\gamma >,\beta \geq 0$ belongs to $T_2^\star(0,\gamma)$ for $z \in E$.*

Proof. For $\beta = 0$, $\tilde{Q}_2(\alpha,0,0,\gamma) = T_2^\star(0,\gamma)$ and the result is obvious. We shall assume that $\beta > 0$.

Form (2.1), we note that, for $f \in \tilde{Q}_2(\alpha,0,0,\gamma)$,

$$f(z) = \phi_{\alpha,\beta}(z) \star F(z),$$

where $F \in T_2^\star(0,\gamma)$ and

$$\phi_{\alpha,\beta}(z) = \sum_{n=1}^{\infty}\left[\frac{(\alpha+\beta)}{\beta(n-1)+\alpha+\beta}\right] z^n.$$

Since $\phi_{\alpha,\beta}(z)$ is convex in E, see [8] and it is known that the class $T_2^\star(0,\gamma)$ is closed under convolution with convex functions [6], we conclude that $f \in T_2^\star(0\gamma)$. □

Using Theorem 2.1 and Theorem 2.5, we can easily show that the class $\tilde{Q}_2(\alpha,\beta,0,\gamma)$ is also closed under convolution with convex functions.

Theorem 2.6. *Let*

$$\frac{\alpha_1}{\alpha_1+\beta_1} < \frac{\alpha}{\alpha+\beta}, \quad \frac{\beta_1}{\alpha_1+\beta_1} < \frac{\beta}{\alpha_1+\beta_1}.$$

Then, for $z \in E$, $\quad \tilde{Q}_2(\alpha, \beta, 0, \gamma) \subset \tilde{Q}_2(\alpha_1, \beta_1, 0, \gamma)$.

Proof. Let $f \in \tilde{Q}_2(\alpha, \beta, 0, \gamma)$. Then , for $z \in E$,

$$\begin{aligned}\frac{\alpha_1}{\alpha_1+\beta_1}\frac{f'(z)}{g'(z)} \quad + \quad \frac{\beta_1}{\alpha_1+\beta_1}\frac{(zf'(z))'}{g'(z)} &= \left(1 - \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}\right)\frac{f'(z)}{g'(z)} \\ &\quad + \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}\left[\frac{\alpha}{\alpha+\beta}\frac{f'(z)}{g'(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{g'(z)}\right] \\ &= \quad 1 - \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}H_1(z) + \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}H_2(z) = H(z),\end{aligned}$$

and since $\tilde{P}(\gamma)$ is a convex set, it follows that $\quad H \in \tilde{P}(\gamma), z \in E$. This implies that $f \in \tilde{Q}_2(\alpha, \beta, 0, \gamma)$. □

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