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THE COMBINATORICS OF DEGENERATE COVERS AND AN APPLICATION FOR GENERAL CURVES OF GENUS 3

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Abstract. Let C_g be a general curve of genus g. If $g \geq 4$ then the monodromy group of a primitive cover $C_g \to \mathbb{P}^1$ of degree n is either S_n or A_n, and both cases actually occur (under suitable conditions on n for fixed g). For $g = 3$ also the groups $GL_3(2)$ and $AGL_3(2)$ occur. In the present paper we settle the last possible case of $AGL_4(2)$. This requires new methods (which may be of independent interest) studying the combinatorial structure of degenerate covers.

1. Introduction

Let C_g be a general curve of genus $g \geq 2$ (over $\mathbb{C}$). Then C_g has a cover to $\mathbb{P}^1$ of degree n if and only if $2(n-1) \geq g$. This is a classical fact of algebraic geometry. If C_g has a cover to $\mathbb{P}^1$ of degree n, then there is such a cover that is simple, i.e., has monodromy group S_n and all inertia groups are generated by transpositions. The question arises whether C_g admits other types of covers to $\mathbb{P}^1$.

If there is a cover $C_g \to \mathbb{P}^1$ branched at r points of $\mathbb{P}^1$ and $g \geq 2$ then $r \geq 3g$ (see Remark 2.2 below). Zariski [Za] used this to show that if $g > 6$ then there is no such cover with solvable monodromy group. The condition $r \geq 3g$ was further used by Guralnick to restrict the possibilities for the monodromy group G of a cover $C_g \to \mathbb{P}^1$ of degree n. Assume the cover does not factor non-trivially, i.e., G is a primitive subgroup of S_n. (Knowledge of this case is sufficient to know all types of covers $C_g \to \mathbb{P}^1$; this was already observed by Zariski [Za], see [GM]). If further

$g > 3$, then $G = S_n$ or $G = A_n$. For $g = 3$ there are 3 additional cases, with $n = 7, 8, 16$ and $G = GL_3(2), AGL_3(2), AGL_4(2)$, respectively. This was proved by Guralnick and Magaard [GM] and Guralnick and Shareshian [GS], using the classification of finite simple groups.

As noted in [GM], it was not known whether the case $G = A_n$ actually occurs. This was answered in the affirmative in [MV]. Also the cases $GL_3(2)$ and $AGL_3(2)$ in genus 3 were settled in [MV]. Here we show that also the last remaining case $G = AGL_4(2)$ occurs in genus 3. This case is more difficult and requires new techniques which may be of independent interest.

Our proof is based on studying degenerations of covers of $\mathbb{P}^1$, i.e., coalescing of branch points. In the usual description using the stable compactification of $\mathcal{M}_{0,r}$, coalescing of branch points means that the lower $\mathbb{P}^1$ degenerates into a tree of genus 0 curves. We describe certain such degenerations of $\mathbb{P}^1$ by the notion of a **multi-list**. The multi-list describes how the branch points are grouped together (in various levels of degeneration) such that the topological model of the degenerate $\mathbb{P}^1$ is obtained by shrinking to a point certain standard paths around blocks of branch points. The points of the degenerate $\mathbb{P}^1$ that arise from the shrinking of such a path are the **nodes**. In our formal approach we actually do not refer to this operation of shrinking paths, but we use the reverse operation of **replacing a node by a tube**.

Recall the usual group-theoretic data associated with a cover of $\mathbb{P}^1$ of degree n: The tuple of branch cycles $\sigma = (\sigma_1, \ldots, \sigma_r)$, where the σ_i are permutations in S_n associated with the branch points (local monodromy). This data depends only on the choice of a homotopy basis of $\mathbb{P}^1$ minus the branch points, and is therefore uniquely determined up to braid group action. Given the degeneration of $\mathbb{P}^1$ (described by a multi-list) and the tuple σ of branch cycles of the original cover of $\mathbb{P}^1$, there is canonically associated a cover of the degenerate $\mathbb{P}^1$. This degenerate cover is constructed recursively in section 3.7. We have transformed this construction into a [GAP4] program which computes the combinatorial structure of this degenerate cover: The genera of the irreducible components, and the way these components are linked together. We further compute the analogous information for the stable model of this covering surface. This is the information actually used in the third part of the paper. We reproduce the GAP code in the appendix of this paper. Thus section 2 is purely topological, extending parts of the usual topological theory of covers of $\mathbb{P}^1$ to the case of covers a tree of $\mathbb{P}^1$'s. Section 2 together with the GAP code in the Appendix is independent of the rest of the paper and may be of interest or usefulness in itself.

In section 3 we complete the proof of our main result by a detailed study of a descending chain of subvarieties in the boundary of the moduli space $\mathcal{M}_3$. These subvarieties classify stable curves of topological type given by the above stable models of covering surfaces.

2. Moduli dimension of a tuple in S_n

2.1. **The Hurwitz space classifying covers of type σ.** Let $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ the Riemann sphere. Let $\mathcal{U}^{(r)}$ be the open subvariety of $(\mathbb{P}^1)^r$ consisting of all $(p_1, ..., p_r)$ with $p_i \neq p_j$ for $i \neq j$, and $\mathcal{U}_r$ the quotient of $\mathcal{U}^{(r)}$ by the action of S_r permuting $p_1, \ldots, p_r$. Thus $\mathcal{U}_r$ is the **configuration space**, consisting of unordered r-tuples of distinct points from $\mathbb{P}^1$. Consider a cover $f : X \to \mathbb{P}^1$ of degree n, with branch points $p_1, ..., p_r \in \mathbb{P}^1$. Pick $p \in \mathbb{P}^1 \setminus \{p_1, ..., p_r\}$, and choose loops γ_i around p_i such that $\gamma_1, ..., \gamma_r$ is a standard generating system of the fundamental group $\Gamma := \pi_1(\mathbb{P}^1 \setminus \{p_1, ..., p_r\}, p)$ (see [V], Thm. 4.27); in particular, we have $\gamma_1 \cdots \gamma_r = 1$. Such a system $\gamma_1, ..., \gamma_r$ is called a homotopy basis of $\mathbb{P}^1 \setminus \{p_1, ..., p_r\}$. The group Γ acts on the fiber $f^{-1}(p)$ by path lifting, inducing a transitive subgroup G of the symmetric group S_n (determined by f up to conjugacy in S_n). It is called the **monodromy group** of f. The images of $\gamma_1, ..., \gamma_r$ in S_n form a tuple of permutations called a tuple of **branch cycles** of f.

Let $\sigma_1, ..., \sigma_r$ be elements $\neq 1$ of the symmetric group S_n with $\sigma_1 \cdots \sigma_r = 1$, generating a transitive subgroup. Let $\sigma = (\sigma_1, ..., \sigma_r)$. We call such a tuple **admissible**. We say a cover $f : X \to \mathbb{P}^1$ of degree n **is of type** σ if it has σ as tuple of branch cycles relative to some homotopy basis of $\mathbb{P}^1$ minus the branch points of f. The genus g of X depends only on σ (by the Riemann-Hurwitz formula); we write $g = g_\sigma$. The **braid orbit** of σ is the smallest set of tuples in S_n that contains σ and is closed under (component-wise) conjugation and under the braid operations

$$(g_1, ..., g_r)^{Q_i} \quad = \quad (g_1, \ldots,\ g_{i+1},\ g_{i+1}^{-1} g_i g_{i+1}\ , \ldots, g_r)$$

for $i = 1, ..., r-1$.

Let $\mathcal{H}_\sigma$ be the set of equivalence classes of covers of type σ. (We use the usual notion of equivalence of covers, see [V], p. 67.) Let σ, σ' be admissible tuples in S_n of length r. Let $f : X \to \mathbb{P}^1$ be a cover of type σ. Then f is of type σ' if and only if σ' lies in the braid orbit of σ. In other words, we have $\mathcal{H}_\sigma = \mathcal{H}_{\sigma'}$ if and only if σ' lies in the braid orbit of σ (see [FrV], [V], Ch. 10).

Let $\Psi_\sigma : \mathcal{H}_\sigma \to \mathcal{U}_r$ be the map that maps the equivalence class of a cover to the set of branch points. The **Hurwitz space** $\mathcal{H}_\sigma$ carries a natural structure of irreducible quasiprojective variety such that Ψ_σ is an algebraic morphism, and an unramified covering in the complex topology (see [FrV],[V], [BeRo]). We also have the morphism

$$\Phi_\sigma :\ \mathcal{H}_\sigma \quad \to \quad \mathcal{M}_g$$

mapping the equivalence class of a cover $f : X \to \mathbb{P}^1$ to the class of X in the moduli space $\mathcal{M}_g$ (where $g = g_\sigma$). Hence the image of Φ_σ, i.e., the locus of genus g curves admitting a cover to $\mathbb{P}^1$ of type σ, is irreducible.

Definition 2.1. *The moduli dimension of σ, denoted by* **mod-dim**(σ)*, is the dimension of the image of Φ_σ; i.e., the dimension of the locus of genus g curves admitting a cover to $\mathbb{P}^1$ of type σ. We say σ has* **full moduli dimension** *if* **mod-dim**$(\sigma) = \dim \mathcal{M}_g$*. Obviously, the moduli dimension of σ depends only on the braid orbit of σ, hence we call it* **the moduli dimension of the braid orbit**.

A curve is called a **general curve of genus** g if it corresponds to a point of $\mathcal{M}_g$ that does not lie in any proper closed subvariety of $\mathcal{M}_g$ defined over $\bar{\mathbb{Q}}$ (the algebraic closure of the rationals). Clearly, an admissible tuple σ has full moduli dimension if and only if each general curve of genus g_σ admits a cover to $\mathbb{P}^1$ of type σ.

The following Remark gives the necessary condition for full moduli dimension used by Guralnick, Fried and Zariski (cf. [MV], Remark 2.2).

Remark 2.2. *Let σ be an admissible tuple of length r in S_n, and $g := g_\sigma \geq 2$. If σ has full moduli dimension then $r \geq 3g$.*

2.2. **Group-theoretic consequences of the necessary criterion for full moduli dimension.** Let $\sigma = (\sigma_1, ..., \sigma_r)$ be an admissible tuple in S_n, and $g := g_\sigma \geq 3$. Assume σ satisfies the necessary condition $r \geq 3g$ for full moduli dimension. Assume further σ generates a primitive subgroup G of S_n. If $g \geq 4$ then $G = S_n$ or $G = A_n$ by [GM] and [GS]. If $g = 3$ and G is not S_n or A_n then one of the following holds (see [GM], Theorem 2):

(1) $n = 7$, $G \cong GL_3(2)$
(2) $n = 8$, $G \cong AGL_3(2)$ (the affine group)
(3) $n = 16$, $G \cong AGL_4(2)$

The affine group $AGL_m(2)$ is the semi-direct product of $GL_m(2)$ with the group of translations. We view it as permutation group on the 2^m points of the affine space $(\mathbb{F}_2)^m$, on which it acts triply transitively. A **transvection** of $AGL_m(2)$ is an involution that fixes a hyperplane of the corresponding affine space pointwise.

In cases (1) and (3), the tuple σ consists of 9 transvections of the respective linear or affine group. In case (2), either σ consists of 10 transvections or it consists of 8 transvections plus an element of order 2,3 or 4 (where the element of order 2 is a translation).

2.3. **Braid orbits of full moduli dimension.**

2.3.1. *Braid orbits of 2-cycle tuples and 3-cycle tuples.* Admissible tuples in S_n of fixed length that consist only of transpositions form a single braid orbit (by Clebsch 1872, see [V], Lemma 10.15). They correspond to the so-called **simple covers**. Their braid orbit has full moduli dimension if and only if $2(n-1) \geq g$, where $g = g_\sigma$ (see the remarks in the Introduction).

Now consider admissible tuples in S_n, $n \geq 6$, of fixed length that consist only of 3-cycles. Such tuples generate A_n. Fried [Fr1] proved that such tuples exist and form exactly two braid orbits (resp., one braid orbit) if $g > 0$ (resp., $g = 0$). In the case $g > 0$, both braid orbits have full moduli dimension by [MV, Theorem 4.1].

It is to be expected that there is a wealth of braid orbits of full moduli dimension whose tuples generate S_n or A_n. A classification seems hopeless.

2.3.2. *Braid orbits of the exceptional tuples in genus 3.* It was proved in [MV, Remark 5.1] that the tuples in case (1) (i.e. 9 double transpositions in S_7 generating a group isomorphic to $GL_3(2)$) form a single braid orbit. This braid orbit has full moduli dimension by [MV, Theorem 5.2].

3. Covers of pinched surfaces

3.1. **Pinched surfaces.** A pinched surface R is a topological space which is obtained from a disjoint union of compact Riemann surfaces $R_1, \ldots, R_s$ by identifying finitely many pairs of points (q_μ, q'_μ) (i.e., we identify q_μ with q'_μ for each μ). These pairs are mutually disjoint. The common image of q_μ and q'_μ in R is denoted by p_μ. We denote the image of R_ν in R by $\bar{R}_\nu$. Each p_μ is contained in at most two $\bar{R}_\nu$. It is allowed that $\bar{R}_\nu$ is linked to itself. The $\bar{R}_\nu$ are called the irreducible components of R, and the p_μ are called the nodes. A node is called to be a node of the **first**, (resp. of the **second**) **kind**, if it lies on exactly one (resp. two) irreducible components of R. A pinched surface is called non-singular if it has no nodes.

3.2. **Replacing a node by a tube.** Let R be a pinched surface and p a node of R. Then p has a neighborhood U that is homeomorphic to the union of two discs D_1 and D_2 that are linked at their midpoints. Let $\tilde{R}$ be the pinched surface obtained by replacing U by a cylinder $\mathcal{T}$ whose two boundary circles coincide with the boundary circles of D_1 and D_2. Obviously, the homeomorphism type of $\tilde{R}$ depends only on R and p. We say $\tilde{R}$ is obtained by replacing the node p by a tube. There is a natural continuous map $\pi : \tilde{R} \to R$ mapping $\pi^{-1}(R \setminus \{p\})$ homeomorphically onto $R \setminus \{p\}$. Furthermore, $\pi^{-1}(p)$ is a circle which we call the **waist-line** of $\mathcal{T}$.

3.3. **The genus of a pinched surface.** We return to the set-up of section 3.1. The **genus** g_ν of the irreducible component $\bar{R}_\nu$ is the genus of the compact Riemann surface R_ν. The **arithmetic genus** g of a connected pinched surface is the genus of the non-singular surface obtained by replacing the nodes by tubes. This genus can be computed from the g_ν by the following formula. Let t be the number of nodes. Then

$$g \;=\; t \;+\; 1 \;+\; \sum_{\nu=1}^{s} (g_\nu - 1) \tag{1}$$

3.4. **Stable pinched surfaces.** An irreducible component of a pinched surface is called **exceptional**, if it has genus 0, is linked to at most two other irreducible components and has no node of the first kind.

A connected pinched surface R of genus $g \geq 2$ is called **stable** if it has no exceptional component. Such a surface of genus $g = 1$ is called stable, if it has no exceptional component and at least one node.

The **stable model** of a pinched surface R of genus ≥ 2 is obtained by repeating the following procedure until we obtain a stable pinched surface: Take an exceptional irreducible component and replace one of its nodes by a tube. The stable model has the same genus.

3.5. **Covers of pinched surfaces.** Let S be a connected pinched surface and $\hat{S}$ the non-singular surface obtained from S by replacing all nodes by tubes. Let $\hat{f} : \hat{R} \to \hat{S}$ be a cover of non-singular surfaces such that no branch point of $\hat{f}$ maps to a node of S. Let $\mathcal{T}$ be a cylinder on $\hat{S}$ coming from a node of S. By our assumption on $\hat{f}$, we may assume that $\mathcal{T}$ contains no branch point of $\hat{f}$. The inverse image of $\mathcal{T}$ in $\hat{R}$ is the disjoint union of cylinders $\mathcal{T}_i$ (because a cylinder is homotopic to a circle). The waist-line $\mathcal{W}$ of $\mathcal{T}$ is homotopic to $\mathcal{T}$, hence each $\mathcal{T}_i$ contains exactly one component $\mathcal{W}_i$ of the inverse image of $\mathcal{W}$. This $\mathcal{W}_i$ is a circle. Shrinking each $\mathcal{W}_i$ to a point results in a pinched surface R. The cover $\hat{f} : \hat{R} \to \hat{S}$ induces a map $f : R \to S$. Each map $R \to S$ obtained in this way is called a cover of pinched surfaces. There is also a direct definition, see [BeRo, Def. 4.4].

Let $\tilde{S}$ be the pinched surface obtained by replacing a single node p of S by a tube. Let $\tilde{R} \to \tilde{S}$ be the cover obtained from $\hat{R} \to \hat{S}$ as in the previous paragraph. In this situation we say that the cover $\tilde{R} \to \tilde{S}$ is obtained from $R \to S$ by **replacing the node p by a tube**.

3.6. **Multi-lists.** Let k be a non-negative integer. A multi-list P of level k is defined as follows: If $k = 0$, then P is a positive integer. If $k > 0$, then $P = (P_1, \ldots, P_t)$, where P_i is a multi-list of level $< k$ and one of the P_i has level $k - 1$.

The integer tuple associated to P is defined as follows: If $k = 0$, then it is the tuple (P). If $k > 0$ and $P = (P_1, \ldots, P_t)$ then the integer tuple associated with P is the concatenation of the integer tuples associated with the P_i. We demand that the integer tuple associated to P is a tuple of consecutive integers.

A multi-list of level 0 is called stable. A multi-list $P = (P_1, \ldots, P_t)$ of level $k > 0$ is called **stable**, if $t \geq 2$ and the P_i are stable for $i = 1, \ldots, t$.

3.7. **The cover associated to a multi-list and a tuple of permutations.** Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ be a tuple of permutations in S_n.

Let $P = (P_1, \ldots, P_t)$ be a stable multi-list of level $k \geq 1$ with associated integer tuple $(m, m+1, \ldots, m')$, where $1 \leq m < m' \leq r$. We define an associated cover of pinched surfaces $R \to S$, where S has genus 0 and carries a distinguished point s_0 which is not a node. This point s_0 is ramified if and only if $\tau \neq 1$, where $\tau = (\sigma_m \cdots \sigma_{m'})^{-1}$.

Let $P_{i_1}, \ldots, P_{i_s}$ be the entries of P of level ≥ 1. For $j = i_1, \ldots, i_s$ let $R^{(j)} \to S^{(j)}$ be the covering associated with the multi-list P_j of level $\leq k-1$ (defined by induction).

Let $S^{(0)}$ be an additional sphere and choose $t+1$ distinct points $p_1, \ldots, p_{t+1}$ on $S^{(0)}$. The last of these points is the distinguished point $s_0 = p_{t+1}$. For $i = 1, \ldots, t$ define $\tau_i = \sigma_k \cdots \sigma_{k'}$, where $(k, k+1, \ldots, k')$ is the integer tuple associated with P_i.

Let $R^{(0)}$ be a cover of $S^{(0)}$ of type $(\tau_1, \ldots, \tau_t, \tau)$ (see [V]), that restricts to an unramified cover of $S^{(0)} \setminus \{p_1, \ldots, p_t, p_{t+1}\}$.

Define the cover $R \to S$ as follows:

(1) The space S is obtained from the disjoint union of the $S^{(j)}$ (for $j = 0, i_1, \ldots, i_s$) by identifying the distinguished point of each $S^{(j)}$ with $p_j \in S^{(0)}$.
(2) The space R is obtained from the disjoint union of the $R^{(j)}$ (for $j = 0, i_1, \ldots, i_s$) by linking $R^{(0)}$ to each $R^{(j)}$, $j = i_1, \ldots, i_s$, in the following way: The points over p_j, on $R^{(0)}$ as well as $R^{(j)}$, correspond to the orbits of τ_j on $\{1, \ldots, n\}$. We identify the points corresponding to the same orbit.

If P has associated integer tuple $(1, \ldots, r)$, then the associated cover $R \to S$ arises from a cover $\hat{R} \to \hat{S}$ of type σ (of non-singular surfaces) as in section 3.2.

Lemma 3.1. *If σ generates a transitive subgroup of S_n, then R is connected.*

Proof. By induction the connected components of $R^{(i)}$ are the orbits of the group H_i, generated by $\sigma_k, \ldots, \sigma_{k'}$, where $(k, k+1, \ldots, k')$ is the integer tuple associated with P_i. The points over p_i correspond to the orbits of τ. Those of these points which lie in the same H_i-orbit belong to the same connected component. As H_i is generated by $\sigma_k, \ldots, \sigma_{k'}$, this shows that the connected components correspond to the orbits of the group generated by all of the σ_i. When this subgroup acts transitive, then R is connected. ∎

Remark 3.2. *We have written a program in* [GAP4] *which computes the combinatorial structure of the covering surface R and its stable model R'. The input of the program is the tuple σ and the multi-list P. The output yields the following information for R as well as R': The genera of the irreducible components, and the links given by nodes of the first and second kind. For the convenience of the reader,* *we reproduce the GAP code in the appendix of this paper. For computing the genera of the components of R and R', we proceed as indicated in the following remark.*

Remark 3.3. *(Genus of the connected surface R)*

Assume σ generates a transitive subgroup of S_n. Then the genus of the connected pinched surface R can be computed by the formula (1) once the genera of the irreducible components of R and the incidence relations are known. The incidence relations can be read off from the algorithm constructing the cover $R \to S$ in section 3.7. The genera can be computed step by step from the following procedure which computes the genera of the components of $R^{(0)}$.

Let H be the subgroup of S_n generated by $\tau_1, \ldots, \tau_t, \tau$. The components of $R^{(0)}$ correspond to the H-orbits on $\{1, \ldots, n\}$. Consider the component Ω corresponding to the H orbit $\mathcal{O}$. Let $\nu_1, \ldots, \nu_{t+1}$ be the restrictions of the generators of H to $\mathcal{O}$. By the Riemann-Hurwitz formula, the genus g_Ω of Ω is given by

$$2(|\mathcal{O}| + g_\Omega - 1) \quad = \quad \sum_{i=1}^{t+1} \mathrm{Ind}(\nu_i)$$

where the index $\mathrm{Ind}(\nu_i)$ is $|\mathcal{O}|$ minus the number of orbits of this permutation.

3.8. **An example in genus 3: the group $AGL_4(2)$.**

3.8.1. *The tuple σ.* Consider the following tuple $\sigma = (\sigma_1, \ldots, \sigma_9)$ in S_{16}, where

$$\begin{aligned}
\sigma_1 &:= (2,6)(3,7)(10,14)(11,15)\\
\sigma_2 &:= (2,6)(3,7)(10,14)(11,15)\\
\sigma_3 &:= (2,7)(3,6)(9,16)(12,13)\\
\sigma_4 &:= (1,3)(6,8)(10,12)(13,15)\\
\sigma_5 &:= (2,7)(4,5)(10,15)(12,13)\\
\sigma_6 &:= (2,16)(4,14)(6,12)(8,10)\\
\sigma_7 &:= (1,13)(3,15)(6,10)(8,12)\\
\sigma_8 &:= (1,10)(2,9)(5,14)(6,13)\\
\sigma_9 &:= (1,15)(2,16)(3,13)(4,14)
\end{aligned}$$

We have $\sigma_1 \cdots \sigma_9 = 1$ and σ generates the group $G = AGL_4(\mathbb{F}_2)$ in its natural action on $\mathbb{F}_2^4$, i.e. on 16 points. By the Riemann-Hurwitz formula we have $g_\sigma = 3$ (cf. section 2.1 and section 2.2).

3.8.2. *The multi-lists P_i.* Consider the multi list

$$P \quad = \quad (((1,2),3),((4,5),6),((7,8),9))$$

The following sequence of multi-lists removes the singularities step by step

$$\begin{aligned}
P_0 &:= (((1,2),3),((4,5),6),((7,8),9))\\
P_1 &:= ((1,2,3),((4,5),6),((7,8),9))\\
P_2 &:= ((1,2,3),(4,5,6),((7,8),9))\\
P_3 &:= ((1,2,3),(4,5,6),(7,8,9))\\
P_4 &:= (1,2,3,(4,5,6),(7,8,9))\\
P_5 &:= (1,2,3,4,5,6,(7,8,9))\\
P_6 &:= (1,2,3,4,5,6,7,8,9)
\end{aligned}$$

3.8.3. *The associated covers.* The construction of section 3.7 associates a sequence of covers of pinched surfaces $R_i \to S_i$, $\quad i = 1, \ldots, 6$. The surface S_i is a tree of $7 - i$ projective lines (i.e., spheres). S_{i+1} arises from S_i by replacing a node by a tube (two spheres of S_i are joined by a tube and thereby merge into one sphere). This implies the corresponding relation between the covering surfaces.

Remark 3.4. *The cover $R_{i+1} \to S_{i+1}$ arises from the cover $R_i \to S_i$ by the process of "replacing a node by a tube" (see section 3.5).*

For $i = 0, \ldots, 6$ let R_i' be the stable model of R_i. All R_i and R_i' have (arithmetic) genus 3 (see section 3.3 and 3.4).

R_0' :	one component of genus 0 with 3 nodes of the second kind linking it to 3 other components (which are mutually disjoint); each of the latter has genus 0 and carries a node of the first kind.
R_1' :	one component of genus 0 with 3 nodes of the second kind linking it to 3 other components (which are mutually disjoint); two of the latter have genus 0 and carry a node of the first kind; the third has genus 1.
R_2' :	one component of genus 0 with 3 nodes of the second kind linking it to 3 other components (which are mutually disjoint); one of the latter has genus 0 and carries a node of the first kind; the two others have genus 1.
R_3' :	three disjoint components of genus 1 linked by three nodes to a component of genus 0
R_4' :	three components of genus 1 linked by two nodes
R_5' :	two components of genus 1, resp. 2, linked by a node
R_6' :	one nonsingular component of genus 3

This information was computed by the program reproduced in the appendix (cf. Remark 3.2). The combinatorial structure of the non-stable covering surfaces R_i is much more complicated. From this it becomes apparent that it would have been extremely tedious to do this computation by hand (although the final result, i.e., the structure of the R_i', is reasonably simple).

We describe the case $i = 0$. The surface R_0 has 58 components of genus 0, the maximal number of nodes on a component is 10. (Note that for any i, the surface R_i has only nodes of the second kind. This is clear from the construction.)

4. The moduli-space of stable curves of genus g

The moduli space $\overline{\mathcal{M}_g}$ classifies stable curves over $\mathbb{C}$ of genus g. It is a projective variety over $\mathbb{C}$. We consider the set of complex points.

4.1. **Covers of pinched surfaces and of algebraic curves.**

Lemma 4.1. *Let $R \to S$ be a cover of pinched surfaces. Let e be a node of S. We replace e by a tube and also all nodes of R which lie over e (as in section 3.2). This gives a covering $R' \to S'$ of pinched surfaces of the same genera. Assume $\mathfrak{R} \to \mathfrak{S}$ is a cover of algebraic curves over $\mathbb{C}$ of topological type $R \to S$. Then there are covers $\mathfrak{R}_t \to \mathfrak{S}_t, t \in [0,1]$ of algebraic curves $/\mathbb{C}$ such that $\mathfrak{R}_t \to \mathfrak{S}_t$ is of topological type $R' \to S'$ for $t \neq 0, |t| < 1$ and the following holds: Let p_t be the point of $\overline{\mathcal{M}_g}$ that corresponds to the stable model of $\mathfrak{R}_t$. Then $p_0 = \lim_{t \to 0} p_t$ in the complex topology.*

Proof: Because of [Man, III.2.7(a),(b), 2.8 (d)], we get a family $\mathfrak{S}_t, t \in [0,1]$, such that $\mathfrak{S}_t$ is of type S for $t = 0$ and type S' for $t > 0$. Because of the specialisation theorem for the Kummerian fundamental group [BeRo, Proposition 7.14] or [AsMaOd], the covering $\mathfrak{R} \to \mathfrak{S}$ deforms into a unique family $\mathfrak{R}_t \to \mathfrak{S}_t, t \in [0,1]$. Then $\mathfrak{R}_t$ is of type R' for $t > 0$.

4.2. **The stratification of $\overline{\mathcal{M}_g}$ by pinched surfaces.** There is a stratification of $\overline{\mathcal{M}_g}$ by the topological type. Let R be a pinched surface of genus g. The stable curves of genus g whose associated pinched surface is homeomorphic to R correspond to the points of a locally closed subset $\overline{\mathcal{M}_g}(R)$ of $\overline{\mathcal{M}_g}$ (see [Man, III.2.8(d)]; that reference uses "modular graphs" [Man, III Definition 2.4] instead of pinched surfaces to describe the topological type of a stable curve).

Let R be a pinched surface of genus g. For any component c of R we define $v(c)$ as the number of nodes on c, with self-intersections counted twice. Then there is a finite morphism

$$\prod_c \mathcal{M}_{g(c),v(c)} \to \overline{\mathcal{M}_g}(R)$$

(see [Man, III.2.8]). This shows that $\overline{\mathcal{M}_g}(R)$ is irreducible and its dimension is given by

$$\dim\overline{\mathcal{M}_g}(R) \quad = \quad \sum_c \,(3g(c) - 3 + v(c))$$

Here we have used $\dim\mathcal{M}_{g,r} = 3g - 3 + r$.

For the genus g we have the formula (1)

$$g \quad = \quad 1 \;+\; \sum_c \,(g(c) - 1) \;+\; 1/2 \sum_c v(c)$$

For $g \geq 2$ this implies

$$\dim\mathcal{M}_g = 3g - 3 = \sum_c (3g(c) - 3 + 3/2 v(c))$$

Corollary 4.2.

$$\mathit{codim}\overline{\mathcal{M}_g}(R) = 1/2 \sum_v c(v) = \textit{number of nodes of } R.$$

Lemma 4.3. [Man, III.2.7(a),(b), 2.8 (d)] *Let R and T be two pinched surfaces. Then $\overline{\mathcal{M}_g}(R)$ is contained in the boundary of $\overline{\mathcal{M}_g}(T)$ if T can be obtained from R by replacing some nodes of R by tubes.*

4.3. **Full moduli dimension for $AGL_4(2)$.** Let $\sigma = (\sigma_1, \ldots, \sigma_9)$, P, R_i and R_i' as in section 3.8. Let $\mathcal{M}^{(i)} := \overline{\mathcal{M}_3}(R_i')$, the locally closed, irreducible subset of $\overline{\mathcal{M}_3}$ classifying stable curves of topological type R_i'. By Corollary 4.2 we have dim $\mathcal{M}^{(i)} = i$, $i = 0, \ldots, 6$.

By inspection we see that R_{i+1}' arises from R_i' by replacing a node by a tube. Therefore, $\mathcal{M}^{(i)}$ is contained in the boundary of $\mathcal{M}^{(i+1)}$ by Lemma 4.3.

Let Ω be the image in $\mathcal{M}_3$ of the Hurwitz space $\mathcal{H}_\sigma$ (see section 2.1). Let $\bar{\Omega}$ be the Zariski-closure of Ω in $\overline{\mathcal{M}_3}$. We want to show $\bar{\Omega} = \overline{\mathcal{M}_3}$.

Lemma 4.4. *Assume $\mathfrak{R} \to \mathfrak{S}$ is a cover of algebraic curves over $\mathbb{C}$ of topological type $R_i \to S_i$ for some $i = 0, \ldots, 6$. Then the stable model of $\mathfrak{R}$ corresponds to a point of $\bar{\Omega}$.*

Proof. For $i = 6$ this follows directly from the definition of Ω. Now assume $i = 5$. By Remark 3.4 and Lemma 4.1, there are covers $\mathfrak{R}_t \to \mathfrak{S}_t, t \in [0,1]$ of algebraic curves $/\mathbb{C}$ such that the following holds: $\mathfrak{R}_t \to \mathfrak{S}_t$ is of topological type $R_6 \to S_6$ for $t \neq 0, |t| < 1$ and equals the given cover $\mathfrak{R} \to \mathfrak{S}$ for $t = 0$. Furthermore, if p_t denotes the point of $\overline{\mathcal{M}_3}$ that corresponds to the stable model of $\mathfrak{R}_t$, then $p_0 = \lim_{t\to 0} p_t$ in the complex topology. This proves the claim for $i = 5$. By iterating this argument we conclude the proof.

Theorem 4.5. *Each general curve of genus 3 has a cover to $\mathbb{P}^1$ with monodromy group $AGL_4(2)$. More precisely, the tuple σ from section 3.8.1 (of nine transvections in $AGL_4(2)$) has full moduli dimension.*

Proof. It suffices to show that $\bar{\Omega} = \overline{\mathcal{M}_3}$ (cf. section 2.1). Recall that dim $\mathcal{M}^{(i)} = i$. By Riemann's Existence Theorem, there is a cover of algebraic curves over $\mathbb{C}$ of topological type $R_0 \to S_0$. It follows by Lemma 4.4 that $\mathcal{M}^{(0)} \subset \bar{\Omega}$. By Lemma 4.1 and because dim $\mathcal{M}^{(1)} = 1$ it follows that there is a Zariski-dense subset $\mathcal{D}$ of points of $\mathcal{M}^{(1)}$ which correspond to the stable model of an algebraic curve $/\mathbb{C}$ covering another algebraic curve of type $R_1 \to S_1$. By Lemma 4.4 we conclude that $\mathcal{M}^{(1)} \cap \bar{\Omega}$ is Zariski-dense in $\mathcal{M}^{(1)}$. It follows that $\mathcal{M}^{(1)} \subset \bar{\Omega}$.

Assume $\mathcal{M}^{(2)}$ is not contained in $\bar{\Omega}$. Then the maximal dimension d of a component of $\mathcal{L} := \mathcal{M}^{(2)} \cap \bar{\Omega}$ satisfies $d < \dim \mathcal{M}^{(2)} = 2$. Since $\mathcal{L}$ is a locally closed subset of $\overline{\mathcal{M}}_3$, each component of the complement of $\mathcal{L}$ in its closure has dimension strictly less than d. Thus the closure of $\mathcal{L}$ would intersect $\mathcal{M}^{(1)}$ in a Zariski-closed proper subset. However, it follows from Lemma 4.1 and Lemma 4.4 that every point of $\mathcal{D}$ lies in the closure of $\mathcal{L}$. This contradiction shows that $\mathcal{M}^{(2)} \subset \bar{\Omega}$.

Continuing like this it finally follows that $\mathcal{M}^{(6)} \subset \bar{\Omega}$. However, $\mathcal{M}^{(6)} = \mathcal{M}_3$, and we are done.

∎

APPENDIX: Computing the combinatorial structure of a (pinched) covering surface given by a tuple of permutations and a multi-list

Appendix A. Auxiliary subroutines

The first of the following subroutines computes the index of a permutation (i.e., the permutation degree minus the number of cycles). The second subroutine computes the genus of any cover of $\mathbb{P}^1$ of type t (cf. section 2.1), where t generates a transitive permutation group of degree n.

```
PermIndex:=function(p,deg)
  return deg - Length(Orbits(Group(p),[1..deg]));
end;


OrbitGenus:=function(t,n)
  if not IsTransitive(Group(t),[1..n]) then Print("Group intransitive");
  return;
    fi;
  return 1-n+ Sum(List([1..Length(t)],
    i->PermIndex(t[i],n)))/2 ;
```

```
end;
```

Appendix B. The combinatorial structure of certain pinched surfaces arising as coverings

Here we transform the recursive construction of the covering $R \to S$ from section 3.7 into a GAP program (cf. Remark 3.2). Let T be an r-tuple of permutations of degree n.

If P is a multi-list with associated integer tuple $1, \dots, r$, the command `IncidMatrix`(n, T, P) produces the following output: A pair whose first entry is the genus of R, where $R \to S$ is the covering constructed from T and P in section 3.7. The second entry is a list I of records, with each record corresponding to a component C of R. The attributes of the record yield the genus of C, and the positions in I of the components linked with C.

The main construction occurs in the subroutine `RecursiveIncidMatrix`, which performs the recursive construction from section 3.7. Most users will not need to call the routine
`RecursiveIncidMatrix`, because it is called automatically by `IncidMatrix`. For completeness, we remark that in `RecursiveIncidMatrix`, the multi-list P is more generally allowed to have integer tuple $(m, m+1, \dots, m')$, where $1 \leq m < m' \leq r$.

```
RecursiveIncidMatrix:=function(n,T,P)
  local dist, G, g, B, record, Perm, Perm1, perm, Orbs, Inc, I, Comp,
  Laengen, ii, i, j, k, l, m, aux, NewOrb, R, s;
  Perm:=[]; Orbs:=[]; Comp:=[]; Laengen:=[];
  if not IsList(P) then return [ T[P], [], [] ];
    fi;
  s:=Length(P);
  for i in [1..s] do
    R:=RecursiveIncidMatrix(n,T,P[i]);
    Add(Perm,R[1]);
    Add(Orbs,R[2]);
    Append(Comp, R[3]); Add(Laengen,Length(R[3]));
    od;
  perm:=Product(Perm);
  Add(Perm, perm^-1);
  G:=Group(Perm);
  NewOrb:= Orbits(G,[1..n]);
  Construction of Comp = list of records, one for each component of the curve
  It has attributes genus and I=list of back distances to incident entries of Comp
  m:=Length(NewOrb);
  for j in [1..m] do
    B:=NewOrb[j];
    g:=1-Length(B)+
      Sum(List([1..s+1],
      i->PermIndex(GeneratorsOfGroup(Action(G,B))[i],Length(B))))/2;
    record:=rec(genus:=g, Inc:= []);
    for i in [1..s] do
```

```
      dist:=0;
      for ii in [i+1..s] do
        dist:=dist+ Laengen[ii];
        od;
      l:=Length(Orbs[i]);
      for k in [1..l] do
        I:=Intersection(NewOrb[j],Orbs[i][k]);
        for aux in Orbits(Group(Perm[i]),I) do
          Add(record.Inc,j+dist+l-k);
          od;
        od;
      od;
    Add(Comp,record);
    od;
  return [perm, NewOrb, Comp];
end;

--------------------------------------------------------------------
IncidMatrix:=function(n,T,P)
  local k, I, II, j, s, r;
  I:=RecursiveIncidMatrix(n,T,P)[3];
  s:=Length(I);
  for j in [1..s] do
    II:= I[j].Inc;
    r:=Length(II);
    for k in [1..r] do
      II[k]:= j-II[k];
      Add(I[II[k]].Inc,j);
```
Incidence relation is made symmetric and absolute (i.e., no relative pointers)
```
      od;
    od;
  return [ OrbitGenus(T,n), I];
end;
```

Appendix C. Computing the stable model of the covering surface

The routine `StabMatrix` has the same input as `IncidMatrix`. It computes the same information with R replaced by its stable model R'.

```
StabMatrix:=function(n,T,P)
  local g, f, flag, k, I, J, II, j, s, m;
  II:= IncidMatrix(n,T,P);
  I:= II[2];
  s:=Length(I);
  flag:=1;
```
now we make the corresponding curve stable if its arithmetic genus is > 1
```
  if II[1]<2 then Print("Curve has genus <2, cannot be made stable");
    fi;
```

```
  while flag=1 do
    flag:=0;
    for j in [1..s] do
      if I[j].genus=0 and Length(I[j].Inc)=1 then
        k:=I[j].Inc[1];
        I[k].Inc:= Filtered(I[k].Inc, x-> not x=j);
        I[j].Inc:=[]; flag:=1;
        fi;
      if I[j].genus=0 and Length(I[j].Inc)=2 then
        k:=I[j].Inc[1];
        m:=I[j].Inc[2];
        I[k].Inc:= Filtered(I[k].Inc, x-> not x=j);
        I[m].Inc:= Filtered(I[m].Inc, x-> not x=j);
        Add(I[k].Inc,m);
        Add(I[m].Inc,k);
        I[j].Inc:=[]; flag:=1;
        fi;
      od;
    od;
  now we delete those components with no incidences left
  and re-label the other components and incidence lists
  J:=[];
  f:=[];
  for j in [1..s] do
    if (not I[j].Inc= []) or I[j].genus>0 then
      Add(J,I[j]);
      f[j]:= Length(J);
      fi;
    od;
  m:=Length(J);
  for k in [1..m] do
    J[k].Inc:= List(J[k].Inc, x-> f[x] );
    od;
  now we compute the arithmetic genus of the stable curve
  as a consistency check (It has to equal II[1]).
  We use the formula in Harris-Morrison, p. 48.
  g:= Sum(List(J, x-> -1 + x.genus + Length(x.Inc)/2)) + 1;
  if g= II[1] then Print(" Arithmetic genus is correct");
    else Print(" Mistake:  Arithmetic genus is wrong");
    fi;
  return [ II[1], J];
end;
```

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