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FULLY INVARIANT τ_M-LIFTING MODULES

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Abstract. Let τ_M be any preradical for $\sigma[M]$ and N any module in $\sigma[M]$. A module N is called *τ_M-lifting* if for every submodule K of N, there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$. We call N is *(strongly) FI-τ_M-lifting* if for every fully invariant submodule K of N, there is a decomposition $K = A \oplus B$, such that A is a (fully invariant) direct summand of N and $B \subseteq \tau_M(N)$. The class of FI-τ_M-lifting modules properly contains the class of τ_M-lifting modules and the class of strongly FI-τ_M-lifting modules. In this paper we investigate whether the class of (strongly) FI-τ_M-lifting modules are closed under particular class of submodules, direct summands and direct sums.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. Let $M \in$ Mod-R. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. For any module M, τ_M will denote a preradical in $\sigma[M]$. We say that A is a *τ_M-coessential submodule of B in N* if $B/A \subseteq \tau_M(N/A)$. Like in [2], a submodule $K \subseteq N$ is called τ_M-supplement (weak τ_M-supplement) provided there exists some $U \subseteq N$ such that $U + K = N$ and $U \cap K \subseteq \tau_M(K)(U \cap K \subseteq \tau_M(N))$. M is called *τ_M-supplemented* (*weakly τ_M-supplemented*) if each of its submodules has a τ_M-supplement (weak τ_M-supplement) in M. M is called *amply τ_M-supplemented*, if for all submodules K and L of N with $K + L = N$, K contains a τ_M-supplement of L in N. A submodule A of N is said to be *τ_M-coclosed in N* if it has no proper τ_M-coessential submodule in N. According to [2] and [12], a module N is called *τ_M-lifting* if for every submodule K of N, there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$. Recall that a submodule K of M is called *fully invariant* (denoted by $K \trianglelefteq M$) if $\lambda(K) \subseteq K$ for all $\lambda \in End_R(M)$.

We mainly study (strongly) FI-τ_M-lifting modules in $\sigma[M]$ in this paper. We call N is *(strongly) FI-τ_M-lifting* if for every fully invariant submodule K of N, there

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is a decomposition $K = A \oplus B$, such that A is a (fully invariant) direct summand of N and $B \subseteq \tau_M(N)$. In Section 1, we show that FI-τ_M-lifting modules are closed under finite direct sums. We prove that if module R_R is FI-τ_M-lifting then R/I has a projective τ_M-cover for every two sided ideal I of R. In Section 2, We show that a direct summand of a strongly FI-τ_M-lifting module is strongly FI-τ_M-lifting and that a finite direct sum of copies of a strongly FI-τ_M-lifting module is strongly FI-τ_M-lifting.

2. FI-τ_M-Lifting Modules

Lemma 2.1. *Let X be a τ_M-supplement submodule of N and $K \subseteq X$. Then X/K is a τ_M-supplement submodule of N/K.*

Proof. See [4, 10.12(3)]. □

Lemma 2.2. *Let M be a module. Then:*

(1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M (in fact the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M).*
(2) *If $X \subseteq Y \subseteq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*
(3) *If $M = \bigoplus_{i \in I} X_i$ and S is a fully invariant submodule of M, then $S = \bigoplus_{i \in I} \pi_i(S) = \bigoplus_{i \in I}(X_i \cap S)$, where π_i is the i-th projection homomorphism of M.*
(4) *If $X \subseteq Y \subseteq M$ such that X is a fully invariant submodule of M and Y/X is a fully invariant submodule of M/X, then Y is a fully invariant submodule of M.*

Proof. (1), (2), (3) See [3, Lemma 1.1]. (4) Let $f : M \to M$ be a homomorphism. Then $f(X) \subseteq X$. Now, consider the homomorphism $g : M/X \to M/X$ defined by $g(m + X) = f(m) + X, (m \in M)$. Then $g(Y/X) \subseteq Y/X$. Clearly, $g(Y/X) = (f(Y) + X)/X$. Therefore $f(Y) \subseteq Y$.
□

We note that if $M = \bigoplus_{i=1}^n M_i$ and N is a fully invariant submodule of M, then $N = \bigoplus_{i=1}^n (N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i.

Lemma 2.3. *Let $N \in \sigma[M]$.The following are equivalent:*

(1) *For every submodule K of N, there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$;*
(2) *For every submodule K of N,there is a direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$;*
(3) *For every submodule K of N, there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$.*

Proof. See [12, Lemma 3.1]. □

A module $N \in \sigma[M]$ is called *τ_M-lifting* if it satisfies one of the equivalent conditions of Lemma 2.3.

Proposition 2.4. *Let $N \in \sigma[M]$. The following are equivalent:*

(1) *For every fully invariant submodule K of N, there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$;*
(2) *For every fully invariant submodule K of N,there is a direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$.*
(3) *For every fully invariant submodule K of N, there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$.*

Proof. (1) $\Rightarrow$ (2) Let K be a fully invariant submodule of N. By hypothesis, there exists a direct summand A of N and $B \subseteq \tau_M(N)$ such that $K = A \oplus B$. Now $N = A \oplus A'$ for some submodule A' of N. Consider the natural epimorphism $\pi : N \to N/A$. Then $\pi(B) = (B + A)/A = K/A \subseteq \tau_M(N/A)$. Therefore N is FI-τ_M-lifting module.

(2) $\Rightarrow$ (3) By [12, Lemma 3.1].

(3) $\Rightarrow$ (1) Let K be a fully invariant submodule of N. By hypothesis, there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$. Therefore $K = A \oplus (K \cap B)$, as required.

□

A module $N \in \sigma[M]$ is called *τ_M-FI-lifting* if it satisfies one of the equivalent conditions of Proposition 2.4.

Theorem 2.5. *Let $N = \bigoplus_{i=1}^n N_i$ be a direct sum of τ_M-FI-lifting modules. Then N is τ_M-FI-lifting.*

Proof. Let $K \trianglelefteq N$. Then $K = \bigoplus_{i=1}^n (K \cap N_i)$ and $K \cap N_i$ is a fully invariant submodule of N_i. As each N_i is τ_M-FI-lifting we have $K \cap N_i = A_i \oplus B_i$ where A_i is a direct summand of N_i and $B_i \subseteq \tau_M(N_i)$. Put $A = \bigoplus_{i=1}^n A_i$ and $B = \bigoplus_{i=1}^n B_i$. Then $K = A \oplus B$ where A is a direct summand of N and $B = \bigoplus_{i=1}^n B_i \subseteq \bigoplus_{i=1}^n \tau_M(N_i) = \tau_M(\bigoplus_{i=1}^n N_i) = \tau_M(N)$. □

Corollary 2.6. *If N is a finite direct sum of τ_M-lifting modules, then N is τ_M-FI-lifting.*

Let $N \in \sigma[M]$. We call an epimorphism $f : P \to N$ a projective τ_M-cover of N in $\sigma[M]$ if P is projective in $\sigma[M]$ and $Ker(f) \subseteq \tau_M(P)$.

Theorem 2.7. *Let P be a projective module. If P is FI-lifting then P/A has a projective τ_M-cover for every fully invariant submodule A of P.*

Proof. Suppose P is a projective FI-lifting module and A is a fully invariant submodule of P. Then $A = X \oplus S$ where X is a direct summand of P and $S \subseteq \tau_M(P)$. Suppose $P = X \oplus Y$. As $S \subseteq \tau_M(P)$, $(X + S)/X \subseteq (X + \tau_M(P))/X \subseteq \tau_M(P/X)$. Hence the natural map $f : P/X \to P/(X + S) = P/A$ is a projective τ_M-cover. □

Corollary 2.8. *Suppose R is a ring. If module R_R is FI-τ_M-lifting then R/I has a projective τ_M-cover for every two sided ideal I of R.*

Proposition 2.9. *Let N be a FI-τ_M-lifting module. Then every fully invariant submodule of $N/\tau_M(N)$ is a direct summand.*

Proof. Let $K/\tau_M(N)$ be a fully invariant submodule of $N/\tau_M(N)$. Then K is fully invariant submodule by Lemma 2.2. By hypothesis, there is a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq K$ and $K \cap N_2 \subseteq \tau_M(N)$. Thus $N/\tau_M(N) = (K/\tau_M(N)) \oplus ((N_2 + \tau_M(N))/\tau_M(N))$, as required.

□

3. Strongly FI-τ_M-lifting Modules

In this section we define strongly FI-τ_M-lifting modules. This class of modules is properly contained in the class of FI-τ_M-lifting modules; but there is no containment relation between the class of strongly FI-τ_M-lifting modules and the class of lifting modules. We show that a direct summand of a strongly FI-τ_M-lifting module is strongly FI-τ_M-lifting and that a finite direct sum of copies of a strongly FI-τ_M-lifting module is strongly FI-τ_M-lifting.

As in Proposition 2.4 we can prove the following.

Proposition 3.1. *Let $N \in \sigma[M]$. The following are equivalent:*

(1) *For every fully invariant submodule K of N, there is a decomposition $K = A \oplus B$, such that A is a fully invariant direct summand of N and $B \subseteq \tau_M(N)$;*

(2) *For every fully invariant submodule K of N,there is a fully invariant direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$.*

A module $N \in \sigma[M]$is called *strongly FI-τ_M-lifting* if it satisfies one of the equivalent conditions of Proposition 3.1.

Proposition 3.2. *Let N be an FI-τ_M-lifting with $\tau_M(N) = 0$. Then every fully invariant submodule (in particular N) is strongly FI-τ_M-lifting module.*

Proof. Let K be a fully invariant submodule of N. Suppose A is fully invariant in K. Then A is fully invariant in N also (see Lemma 2.2). As N is FI-τ_M-lifting, $A = B \oplus S$ where B is a direct summand of N and $S \subseteq \tau_M(N)$ (see Proposition 2.4). Since $\tau_M(N) = 0$, $S = 0$ and so A is a direct summand of N and hence of K. Thus K is strongly FI-lifting. □

Theorem 3.3. *A direct summand of a strongly FI-τ_M-lifting module is strongly FI-τ_M-lifting.*

Proof. Let $N = X \oplus Y$ be a strongly FI-τ_M-lifting module. Assume that $S_1 \trianglelefteq X$. Then there exists $S_2 \trianglelefteq Y$ such that $S_1 \oplus S_2 \trianglelefteq M$ [5, Lemma 1.11]. Since N is a strongly FI-τ_M-lifting, $S_1 \oplus S_2 = B \oplus S$ where $S \subseteq \tau_M(N)$ and B is a fully invariant direct summand of N. But $B \trianglelefteq N$ implies that $B = (X \cap B) \oplus (Y \cap B)$ and $X \cap B$ is fully invariant in X. Also $X \cap B$ is a direct summand of N. We have $S_1 = \pi_X(B) + \pi_X(S) = (X \cap B) + \pi_X(S)$ where $\pi_X : N \to X$ is the projection along Y. As $S \subseteq \tau_M(N), \pi_X(S) \subseteq \tau_M(X)$. By Proposition 3.1, X is a strongly FI-τ_M-lifting module. □

Proposition 3.4. *Let $N = \bigoplus_{i=1}^n N_i$ and let $N_i \trianglelefteq M$ for all $1 \leq i \leq n$. Then N is strongly FI-τ_M-lifting if and only if N_i is strongly FI-τ_M-lifting, for all $1 \leq i \leq n$.*

Proof. If N is strongly FI-τ_M-lifting then each N_i is so, by Proposition 3.4.

Conversely, suppose $N = \bigoplus_{i=1}^n N_i$ where each N_i is strongly FI-τ_M-lifting and fully invariant in N. Let $K \trianglelefteq N$. Then $K = \bigoplus_{i=1}^n (K \cap N_i)$ and $(K \cap N_i) \trianglelefteq N_i$, for all $1 \leq i \leq n$. As N_i is strongly FI-τ_M-lifting, $K \cap N_i = B_i \oplus S_i$ where B_i is a fully invariant direct summand of N_i and $S_i \subseteq \tau_M(N_i)$ (see Proposition 3.1). Put $B = \bigoplus_{i=1}^n B_i$ and $S = \bigoplus_{i=1}^n S_i$. Then $K = B \oplus S$ where B is a direct summand of N and $S \subseteq \tau_M(N)$. As $B_i \trianglelefteq N_i$ and $N_i \trianglelefteq N$, $B_i \trianglelefteq N$ for all $1 \leq i \leq n$. Hence $B \trianglelefteq N$. Therefore N is strongly FI-τ_M-lifting. □

Theorem 3.5. *Suppose K is a strongly FI-τ_M-lifting module and $N = \bigoplus_{i=1}^{n} N_i$ where each $N_i \simeq K$. Then N is a strongly FI-τ_M-lifting module.*

Proof. There exist isomorphisms $f_i : N_1 \to N_i$ for $i = 2, \cdots, n$. If A is a fully invariant submodule of N, then it is easy to see that $A = A_1 \oplus f_2(A_1) \oplus \cdots \oplus f_n(A_1)$ where $A_1 = N_1 \cap A$.

As N_1 is strongly FI-τ_M-Lifting and A_1 is a fully invariant submodule of N_1, we have $A_1 = L_1 \oplus S_1$ where L_1 is a fully invariant submodule of N_1 and $S_1 \subseteq \tau_M(M_1)$ (see Proposition 3.1). Put $L := L_1 \oplus f_2(L_1) \oplus \cdots \oplus f_n(L_1)$ and $S := S_1 \oplus f_2(S_1) \oplus \cdots \oplus f_n(S_1)$. Then $A = L \oplus S$, L is a fully invariant direct summand of N and $S \subseteq \tau_M(N)$. Hence N is strongly FI-τ_M-lifting. □

From Proposition 3.3 and Theorem 3.5 we get the following.

Corollary 3.6. *Suppose R is a ring and R_R is strongly FI-τ_M-lifting. Then any finitely generated projective R-module is strongly FI-τ_M-lifting.*

Proposition 3.7. *Let N be a strongly FI-τ_M-lifting module and X a fully invariant submodule of N. If X is indecomposable, then X is strongly FI-τ_M-lifting.*

Proof. Let $K \trianglelefteq X$ then $K \trianglelefteq N$. Since N is strongly FI-τ_M-lifting module, $K = B \oplus S$ where $S \subseteq \tau_M(N)$ and B is a fully invariant direct summand of N. Hence $N = B \oplus C$ for some submodule C of N. Since $X \trianglelefteq N$, then $X = B \oplus (C \cap X)$. But X is indecomposable, therefore $X = B$ and X is direct summand of N. By Theorem 3.3, X is strongly FI-τ_M-lifting. □

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