ALBANIAN JOURNAL
OF MATHEMATICS
Volume 3, Number 1, Pages 87–94
ISSN 1930-1235: (2009)

AUXILIARY PRINCIPLE TECHNIQUE FOR NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and analyze some iterative methods for solving nonconvex variational inequalities using the auxiliary principle technique, the convergence of these methods either requires only pseudomonotonicity or partially relaxed strongly monotonicity. Our proofs of convergence are very simple. As special cases, we obtain earlier known results for solving variational inequalities involving the convex sets.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [30], can be viewed as an important and significant extension of the variational principles, the origin of which can be traced back to Fermat, Bernoulli brother, Euler, Lagrange. This provides us with a simple, general and unified framework to study a wide class of problems arising in pure and applied sciences. This theory combines the theory of extremal problems and monotone operators under a unified viewpoint. It is perhaps part of the fascinating of this theory that so many branches of pure and applied sciences are involved. During the last five decades, there has been considerable activity in the development of numerical techniques for solving variational inequalities. There are a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle, and descent framework for solving variational inequalities and complementarity problems; see [1,2,4-28]. It is worth mentioning that almost all the results regarding the existence and iterative schemes for solving variational inequalities and related optimization problems are being considered in the convexity setting. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. In recent years, Noor [13, 18-21,24], Bounkhel et al [2] and Pang et al [28] have considered variational inequality in the context of uniformly prox-regular sets. They have shown that the nonconvex variational inequalities are equivalent to the fixed point problems using the projection techniques. They have used this alternative equivalent formulation to suggest and analyze some projection-type iterative schemes for solving nonconvex variational inequalities. It has been shown that the convergence of these projection-type methods requires that the operator must be both strongly

Received by the editors April 2, 2009 .
2000 *Mathematics Subject Classification.* Primary 49J40; Secondary 90C33.
Key words and phrases. Variational inequalities; nonconvex functions; auxiliary principle technique, convergence.

monotone and Lipschitz continuous. These strict conditions rule out many its applications. Secondly it is very difficult to evaluate the projection of the space onto the uniformly prox-regular sets. To overcome this drawback, we use the auxiliary principle technique, which is mainly due to Glowinski, Lions and Tremolieres [5]. Noor [10-15, 24] has used this technique to develop some iterative schemes for solving various classes of variational inequalities. We point out that this technique does not involve the projection of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be used to suggest and analyze a class of iterative methods for solving nonconvex variational inequalities. We also prove that the convergence of these new methods either require pseudomonotonicity or partially relaxed strongly monotonicity, which are weaker conditions. In this respect, our results represent an improvement and refinement of the known results for nonconvex variational inequalities.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H.

We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [3,29].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempt, closed and convex subset in H. Then $\zeta \in N_K^P(u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. The Clarke normal cone, denoted by $N_K^C(u)$, is defined as

$$N_K^P(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^P(u)$ is always closed and convex, whereas $N_K^C(u)$ is convex, but may not be closed, see [29].

Poliquin et al. [29] and Clarke et al [3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.3. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u \rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3,29]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N^P_{K_r}(u)$ is closed as a set-valued mapping.

For a given nonlinear operator T, we consider the problem of finding $u \in K_r$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K_r, \tag{1}$$

which is called the *nonconvex variational inequality,* which was introduced and studied by Noor [18-21,24]. See also [2, 27] for the variant forms of nonconvex variational inequalities.

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \forall v \in K. \tag{2}$$

Inequality of type (2) is called the *variational inequality*, which was introduced and studied by Stampacchia [30] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1-30] and the references therein.

It is well-known that problem (2) is equivalent to finding $u \in K$ such that

$$0 \in Tu + N_K(u), \tag{3}$$

where $N_K(u)$ denotes the normal cone of K at u in the sense of convex analysis. Problem (3) is called the variational inclusion associated with variational inequality (2).

Similarly, if K_r is a nonconvex (uniformly prox-regular) set, then problem (1) is equivalent to finding $u \in K_r$ such that

$$0 \in Tu + N^P_{K_r}(u), \tag{4}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. Problem (4) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (1). This implies that the variational inequality (1) is equivalent to finding a zero of the sum of two monotone operators (4).

3. Main Results

In this section, we use the auxiliary principle technique of Glowinski, Lions and Tremolieres [5] to suggest and analyze a some iterative methods for solving the nonconvex variational inequality (1). The main advantage of this technique does not involve the concept of the projection, which is the main advantage of this technique.

For a given $u \in K_r$, a uniformly prox-regular set in H, consider the problem of finding a u solution $w \in K_r$ such that

$$\langle \rho Tw + w - u, v - w\rangle \geq 0, \quad \forall v \in K_r, \tag{5}$$

where $\rho > 0$ is a constant. Inequality of type (5) is called the auxiliary nonconvex variational inequality. Note that if $w = u$, then w is a solution of (1). This simple observation enables us to suggest the following iterative method for solving the nonconvex variational inequalities (1).

Algorithm 3.1. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K_r. \tag{6}$$

Algorithm 3.1 is called the proximal point algorithm for solving noconvex variational inequality (1). In particular, if $r = \infty$, then the uniformly prox-regular set K_r becomes the standard convex set K, and consequently Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K,$$

which is known as the proximal point algorithm for solving variational inequalities (2) and has been studied extensively, see [2,4-27].

For the convergence analysis of Algorithm 3.1, we recall the following concepts and results.

Definition 3.1. For all $u, v, z \in H$, an operator $T : H \to H$ is said to be:
(i) *monotone*, if

$$\langle Tu - Tv, u - v \rangle \geq 0.$$

(ii) *pseudomonotone*, if

$$\langle Tu, v - u \rangle \geq 0 \quad \text{implies that} \quad \langle Tv, u - v \rangle \leq 0.$$

(iii) *partially relaxed strongly monotone,* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z - v \rangle \geq -\alpha \|z - u\|^2.$$

Note that for $z = u$, partially relaxed strongly monotonicity reduces to monotonicity. It is known that cocoercivity implies partially relaxed strongly monotonicity, but the converse is not true. It is known that monotonicity implies pseudomonotonicity; but the converse is not true. Consequently, the class of pseudomonotone operators is bigger than the one of monotone operators.

Lemma 3.1. $\forall u, v \in H$,

$$2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{7}$$

We now consider the convergence criteria of Algorithm 3.1. The analysis is in the spirit of Noor [13,14,15,24].

Theorem 3.1. Let the operator $T : K_r \longrightarrow H$ be pseudomonotone. If u_{n+1} is the approximate solution obtained from Algorithm 3.2 and $u \in K_r$ is a solution of (1), then

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{8}$$

Proof. Let $u \in K_r$ be a solution of (1). Then

$$\langle Tv, v - u \rangle \geq 0, \quad \forall v \in K_r, \tag{9}$$

since T is pseudomonotone.

Taking $v = u_{n+1}$ in (5), we have

$$\langle Tu_{n+1}, u_{n+1} - u\rangle \geq 0. \tag{10}$$

Setting $v = u$ in (6), and using (10), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq \rho\langle Tu_{n+1}, u_{n+1} - u\rangle \geq 0. \tag{11}$$

Setting $v = u - u_{n+1}$ and $u = u_{n+1} - u_n$ in (7), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2 - \|u - u_{n+1}\|^2. \tag{12}$$

From (11) and (12), we obtain (8), the required result. □

Theorem 3.2. Let H be a finite dimension subspace and let u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $u \in K_1$ is a solution of (1), then $\lim_{n\longrightarrow\infty} u_n = u$.

Proof. Let $u \in K_r$ be a solution of (1). Then it follows from (8) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_n - u_{n+1}\| = 0. \tag{13}$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K_r$. replacing u_n by u_{n_j} in (6) and taking the limit $n_j \longrightarrow \infty$ and using (13), we have

$$\langle T\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall \quad v \in K_r,$$

which implies that $\hat{u}$ solves the nonconvex variational inequality (1) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. the required result. □

We note that for $r = \infty$, the r-prox-regular set K becomes a convex set and nonconvex variational inequality (1) collapses to variational inequality (2). Thus our results include the previous known results as special cases.

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in K_r$, consider the problem of finding $w \in K_r$ such that

$$\langle \rho Tu + w - u, v - w\rangle \geq 0, \quad \forall \quad v \in K_r, \tag{14}$$

which is also called the auxiliary variational inequality. Note that problems (6) and (14) are quite different. If $w = u$, then clearly w is a solution of the nonconvex variational inequality (1). This fact enables us to suggest and analyze the following iterative method for solving the nonconvex variational inequality (1).

Algorithm 3.3. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K_r. \tag{15}$$

Note that for $r = \infty$, the uniformly prox-regular set K_r becomes a convex set K and Algorithm 3.3 reduces to:

Algorithm 3.4. For a given $u_0 \in K$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K,$$

or equivalently

$$u_{n+1} = P_K[u_n - \rho T u_n], \quad n = 0, 1, 2, \ldots,$$

which is known as the projection iterative method for solving convex variational inequalities (2) and have been studied extensively.

We now study the convergence of Algorithm 3.3 and this is the main motivation of our next result.

Theorem 3.3. Let the operator T be partially relaxed strongly monotone with constant $\alpha > 0$. If u_{n+1} is the approximate solution obtained from Algorithm 3.3 and $u \in K_r$ is a solution of (1), then

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \{1 - 2\rho\alpha\}\|u_n - u_{n+1}\|^2. \tag{16}$$

Proof. Let $u \in K_r$ be a solution of (1). Then

$$\langle Tu, v - u \rangle \geq 0, \quad \forall \quad v \in K_r. \tag{17}$$

Taking $v = u_{n+1}$ in (17), we have

$$\langle Tu, u_{n+1} - u \rangle \geq 0. \tag{18}$$

Letting $v = u$ in (15), we obtain

$$\langle \rho T u_n + u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle u_{n+1} - u_n, u - u_{n+1} \rangle &\geq \langle \rho T u_n, u_{n+1} - u \rangle \\ &\geq \rho \langle T u_n - Tu, u_{n+1} - u \rangle \\ &\geq -\alpha\rho \|u_n - u_{n+1}\|^2. \end{aligned} \tag{19}$$

since T is partially relaxed strongly monotone with constant $\alpha > 0$.

Combining (19) and (12), we obtain the required result (16). □

Using essentially the technique of Theorem 3.2, one can study the convergence analysis of Algorithm 3.3.

Acknowledgement. The author would like to express his gratitude to Dr. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities.

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