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MORSE THEORY WITH LOW DIFFERENTIABILITY AND DEGENERATE CRITICAL POINTS FOR FUNCTIONAL ENERGY OF A FINSLER METRIC

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Abstract. The aim of this paper is to extend the Morse theory of $(\Lambda M, E)$ with low differentiability and degenerate critical points, where ΛM is the space of H^1-closed curves on an n-dimensional compact manifold M endowed with a Finsler metric $F : TM \to R$ and $E : \Lambda M \to R$ is the associated energy integral, or simply the energy.

1. Introduction

In the Morse theory with low differentiability and degenerate critical points, on Hilbert manifolds, the closed geodesics problem for Finsler metric can be developed as in the Riemannian case; see [1], [7]. In this theory a closed geodesic is a distinguished closed curve in the Hilbert manifold of H^1-closed curves and being a critical point of the functional integral energy of Finler metric F. The other aspect is to consider a closed geodesic (or more exactly the tangent vector field along a closed geodesic) as a periodic orbit in the geodesic flow on the cotangent bundle T^*M of a Finsler manifold $(M\,,\,F)$. In this case the geodesic flow is a special case of a Hamiltonian flow (see, [8], [23], [24]). One of the differences between the Morse theory for the Finsler and Riemannian cases is that the Finsler energy is not C^2. In fact it is twice differentiable at the critical points, and with strongly differentiable derivative on these points, but not, in general, outside the regular

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closed curves (see, Theorem 8.1 of this paper). This is a peculiarity of the Finsler metrics since the energy E is C^2 if only if F is the norm of a Riemannian metric. Therefore, in order to have a Morse theory for the Finsler case we need a Morse lemma for functions with conditions of low differentiability that, although stronger than those in [10], are verified by the Finsler energy. This Morse lemma was done in the papers (see [1], [7]) for the case of isolated critical points of functions with low differentiability and in a very general context that beyond allowing the adaptation of Gromoll–Meyer arguments to our case, has probable utility in the treatment of more general variational problems. A very interesting and important question is to extend the Gromoll–Meyer theorem for Finsler metrics. In fact this is the case, and the first demonstration is due to Matthias; see [13]. The Matthias demonstration used an approach of finite dimension of the Morse theory for closed geodesics of the manifold of the H^1- closed curves, theory inspired by the treatment of Milnor (see [17]) of the problem of geodesics connecting two points. This type of argument, elegant and extremely efficient in the cases above, uses in an essential way the geometry of the metric and does not seem to be appropriate to the study of other more general 1-dimensional variational problems.

The purpose of this paper is to extend some results of the theory of critical points that allow to use of the original arguments of Gromoll–Meyer to prove the existence of infinitely many closed geodesics (non trivial and geometrically distinct) on a compact simply connected differentiable Finsler manifold whose cohomology ring is not generated by only one element (i.e., the cohomology ring is not isomorphic to the one of a compact symmetric space of rank one: a sphere or a complex projective space, quaternionic or of Cayley), provided that the sequence of the rational Betti numbers of the space ΛM of parametrized H^1-closed curves is unbounded. The Morse lemma that we proved in the papers [1] and [7] for degenerate critical points with low differentiability, is for a C^1- function defined on an open neighborhood of 0 in a Hilbert space H, where 0 is the only critical point of f, f twice differentiable at 0 with derivative f' strongly differentiable at the origin. This result generalizes the Morse lemma for a nondegenerate critical point of C^2 function that is due to Cambini (see [2]) and also the Morse lemma for degenerate case for a C^2 function that was extended by Mawhin and Willem; see [15], [16].

We extend to compact connected critical submanifold of a Hilbert manifold the notion of critical point and we compute the critical groups at an isolated critical submanifold of the Finsler energy E. In the paper [9], F. Mercuri describes the Liusternik–Schnirelman theory for the energy E of a Finsler metric F on a smooth manifold M, and the following properties are proved:

a. The energy E is C^1 and the differential is locally Lipschitzian.
b. If M is compact, E satisfy the condition (C) of Palais–Smale.

The our main purpose in this paper is to prove the following:

(1) Under the hypothesis of Proposition 3.2 (A Morse lemma for degenerate critical points with low differentiability), we prove that the function $\hat{f}$ associated to the degenerate part of the function f, is of class C^1, with derivative $(\hat{f})'$ strongly differentiable at $y = 0$ with $(\hat{f})''(0) = 0$, and also we compute the local critical groups.
(2) The derivative of the Finsler energy $E : \Lambda M \to \mathbb{R}$ is strongly differentiable on the regular curves and particularly on the closed geodesics, and therefore

E satisfies the hypothesis of the our Morse lemma for isolated critical points, possibly degenerate (Proposition 3.2, page 7).

(3) To generalize the Gromoll–Meyer theorem for Riemannian manifolds to Finsler manifolds.

A more direct approach using the Morse theory on Hilbert manifold now is possible since the energy functional E in Finsler case satisfies the hypothesis of our Morse lemma for degenerate critical point with low differentiability. Moreover, the index formula for Finsler manifolds works the same in the Riemannian case. The index form of the energy functional E in the Finsler case can be found already explicitly in standard text books and papers; see [7], [12], [25]. This index theory for closed geodesics on Finsler manifolds with our previous results permit the conclusion of the existence of infinitely many closed geodesics on a compact Finsler manifold, whose cohomology is not isomorphic to that of a compact symmetric space of rank one.

2. Preliminaries

Let M a Hilbert manifold and $f \in C^{2-}(M,\mathbb{R})$, i. e. f is C^1-function with differential df locally Lipschitzian. We assume that the manifold M is regular, i. e. every neighborhood of a point contains a closed neighborhood. Let u be an isolated critical point of f. The critical groups (over a field F) are defined by

$$C_n(f,u) = H_n(\{f \le c\} \cap U\,,\, \{f \le c\} \cap U - \{u\}), \quad n = 0,1,2,\dots,$$

where $c = f(u)$,

$$\{f \le c\} = \{v \in M \,:\, f(v) \le c\},$$

U is a closed neighborhood of u, and $H_n(A\,,\,B)$ is the n-th singular homology group of the pair $(A\,,\,B)$ over a field F. By excision, the critical groups are independent of U. We recall that f satisfies the condition (C) of Palais–Smale over a closed subset S of M if, for any sequence z_n on S such that $f(z_n)$ is bounded and $|f'(z_n)|$ tends to zero, then (z_n) has a convergent subsequence and any limit point is a critical point of f. It is easy to see that a C^1 function f, twice differentiable at critical points that we suppose isolated, and admitting a nondegenerate critical point, always satisfies the condition (C) in a closed neighborhood of that point. A piecewise C^1 path from u_1 to u_2 is a piecewice C^1 mapping

$$\sigma \,:\, [a\,,\,b] \to M$$

that $\sigma(a) = u_1$ and $\sigma(b) = u_2$. We define the length of σ by

$$L(\sigma) = \int_a^b |\sigma'(t)|\,dt,$$

and the geodesic distance d on M by the

$$d(u_1,u_2) = inf\{L(\sigma);\ \sigma : [a,b] \to M,\ \sigma(a) = u_1,\ \sigma(b) = u_2,\ \sigma \text{ a piecewise } C^1\}.$$

A subset of M is complete if it is complete for geodesic distance. The gradient of f is the vector field ∇f defined on M by

$$df(u)v = \langle \nabla f(u)\,,\, v\rangle \ \ \forall v \in T_uM.$$

Proposition 2.1. *Let $f \in C^{2-}(M,\mathbb{R})$ and $\sigma(t) = \sigma(t,u),\ \alpha(u) \le t \le \beta(u)$, the unique maximal solution of the equation $\dot\sigma(t) = -\nabla f(\sigma(t)),\ \sigma(0) = u$. Then the following conditions are true:*

(1) *Either* $f(\sigma(t)) = f(u)$ *or* $f \circ \sigma$ *is non-increasing for all* $t \geq 0$. *Moreover, for* $0 \leq s \leq t < \beta(u)$, *we have*

$$d(\sigma(t), \sigma(s)) \leq L(\sigma) \leq (t-s)^{1/2}(f(\sigma(s)) - f(\sigma(t)))^{1/2}.$$

(2) *If* $\beta(u)$ *is finite and the set* $\{\sigma(t) : \ 0 \leq t < \beta(u)\}$ *is contained in a complete subset of* M, *then* $f(\sigma(t)) \to -\infty$ *when* $t \to \beta(u)$.

(3) *If the set* $\{\sigma(t) : \ 0 \leq t < \beta(u)\}$ *is contained in a complete subset of* M *with* $f(\sigma(t)) > a$, *then* $\beta(u) = +\infty$.

3. Strongly Differentiable Functions

Recall that a function between two Banach spaces, $f : E \to F$, is said to be strongly differentiable at a when there is a linear map $T : E \to F$, such that for $x, y \in E$:

$$f(x) - f(y) \, = \, T(x-y) + R(x,y),$$

where $\lim\limits_{x,\, y \to a} \frac{R(x,y)}{\|x-y\|} = 0$.

Taking $y = a$, we see that a strongly differentiable function f at a is differentiable at a, with $T = f'(a)$. Another way to be equivalent to define a strongly differentiable function is as follows.

The function $f : E \to F$, is said to be *strongly differentiable* at a, if f is differentiable at a and $\lim\limits_{x,\, y \to a} \frac{r(x)-r(y)}{\|x-y\|} = 0$, where $r(y)$ is the remainder of the Taylor's formula for f around a.

In other words, f is strongly differentiable at a, if and only if, it is differentiable at a and, given $\varepsilon > 0$, there is a neighborhood $V$of a where $r(x)$ is $\varepsilon-Lipschitzian$, and therefore f satisfies the condition of Lipschitz, with constant $\|f'(a)\| + \varepsilon$.

It is clear also that if f is differentiable in a neighborhood of a and its differential f' is continuous at a, then f is strongly differentiable at a. Moreover, if f is continuous in a neighborhood of a and strongly differentiable at a, with invertible differential, then f is invertible around a. The proof is the same as the classical inverse function theorem; see [6, chapter 5].

Let $f : U \subset H \to \mathbb{R}$ be a C^1 function defined on a open set of a Hilbert space H and $0 \in U$. Suppose that f is twice differentiable at 0, having 0 as critical point. Let N be the kernel of the symmetric operator $A : \ H \to H$ given by

$$\langle Av, u \rangle \, = \, \frac{1}{2} d^2 f(0)(v, u).$$

If the image $Im(A)$ is closed, since A is symmetric, $N^\perp = Im(A)$ and H is decomposable in $H = N^\perp \oplus N$. Thus we can look at $z \in H$ as $x + y \in N^\perp \oplus N$. Also the corresponding version of the implicit function theorem gives the following proposition.

Proposition 3.1. *Using the conditions and notation above, let* 0 *be a critical point of* f *and suppose that* f' *is strongly differentiable at the origin. Then there is a continuous function*

$$g : \, U \subset N \to N^\perp$$

on an open set U *containing* 0 *such that* $f_x(g(y), y) \equiv 0$ *and* $g(0) = 0$, *where* f_x *denote the partial derivative with respect to* x. *Moreover,* g *is strongly differentiable at the origin and* $\mathrm{d}g(0) = 0$.

In the paper [1] we proved a degenerate-critical point version of the Morse lemma as in [4] with conditions of low differentiability that, although stronger than those in [9], are verified by the Finsler energy $E : \Lambda M \to \mathbb{R}$. These results are contained as part of the author PhD disertation; see [7].

Proposition 3.2 (A Morse Lemma for degenerate critical points with low differentiability)**.** *If f' is strongly differentiable at the origin, then there is a neighborhood V of 0 in H and a homeomorphism $\psi : V \to \psi(V) \subset H$, $\psi(0) = 0$ such that*

$$f(\psi(x, y)) \,=\, \frac{1}{2}\langle Ax, x\rangle \,+\, f(g(y), y),$$

where g is a function $g : V \bigcap N \to N^{\perp}$ strongly differentiable at $0 \in N$ with $g(0) = 0$, and $\mathrm{d}g(0) = 0$, the function ψ is differentiable at 0, with $\psi(0) = 0$ and $d\psi(0) = I$; see [7] and [1].

The next proposition that we will prove is very important for the computation of the local critical groups at a isolated critical point, that is reduced to a finite dimension problem.

Proposition 3.3. *Let $f : U \subset H \to R$ be a C^1 function defined on a open set and let $0 \in U$ be only critical point of f. Suppose that f is twice differentiable at 0 and that f' is strongly differentiable at the origin. Let*

$$z = x + y \in H = N^{\perp} \oplus N$$

and f_x, f_y be the partial derivatives with respect to x and y, and

$$g : B_r(0) \subset N \to N^{\perp}$$

be a unique continuous map defined on an open ball $B_r(0) = \{y \in N : |y| < r\}$, such that

$$f_x(g(y), y) \equiv 0,$$

g is strongly differentiable at the origin with $g(0) = 0$, $dg(0) = 0$ and g is Lipschitzian on $B_r(0)$. Then, the following hold

1. *The function $\hat{f} : B_r(0) \subset N \to R, \hat{f}(y) = f(g(y), y)$ is of class C^1 with $(\hat{f})'(y) = f_y(g(y), y)$ where $0 \in B_r(0)$ is only critical point of $\hat{f}$ and $(\hat{f})'$ is Lipschitzian on $B_r(0)$.*
2. *If f satisfies the condition (C) of Palais–Smale, then $\hat{f}$ also satisfies this condition.*
3. *The function $\hat{f} : B_r(0) \subset N \to R$, $\hat{f}(y) = f(g(y), y)$ is twice differentiable at the origin with $(\hat{f})''(0) = 0$, and the derivative $(\hat{f})'$ is strongly differentiable at $y = 0$.*

Proof. We can choose the open set $U \subset H$ a convex open set

$$U = B_\delta(0) \oplus B_r(0),$$

with g Lipschitzian on $B_r(0)$, f_x *and* f_y Lipschitzian on $B_\delta(0) \oplus B_r(0)$. Let $M_1 > 0$, $M_2 > 0$, $M_3 > 0$ be the constants of Lipschitz for $g : B_r(0) \to B_\delta(0)$, $f_x : B_\delta(0) \oplus B_r(0) \to R$ *and* $f_y : B_\delta(0) \oplus B_r(0) \to R$, respectively.

Furthermore, let y_0, $y_0 + h \in B_r(0)$ with h sufficiently small and $f_x(g(y_0, y_0)) = f_x(g(y_0 + h), y_0 + h) \equiv 0$ and $r(h) = \hat{f}(y_0 + h) - \hat{f}(y_0) - f_y(g(y_0), y_0)h$ be the remainder of Taylor's formula. Then, we have

$$
\begin{aligned}
|r(h)| &= |f(g(y_0+h), y_0+h) - f(g(y_0, y_0)) - f_y(g(y_0), y_0)h| \\
&\leq |f(g(y_0+h), y_0+h) - f(g(y_0), y_0+h)| \\
&\quad +|f(g(y_0), y_0+h) - f(g(y_0, y_0) - f_y(g(y_0), y_0)h| \\
&\leq |g(y_0+h) - g(y_0)| \sup_{0\leq t\leq 1} |f_x(g(y_0) + t(g(y_0+h) - g(y_0)), y_0+h| \\
&\quad +|h| \sup_{0\leq t\leq 1} |f_y(g(y_0), y_0+th) - f_y(g(y_0), y_0)| \\
&\leq M_1|h| \sup_{0\leq t\leq 1} |f_x(g(y_0) + t(g(y_0+h) - g(y_0)), y_0+h) - f_x(g(y_0), y_0)| \\
&\quad +|h| \sup_{0\leq t\leq 1} |f_y(g(y_0), y_0+th) - f_y(g(y_0), y_0)| \\
&\leq M_1M_2|h| \sup_{0\leq t\leq 1} |(t(g(y_0+h) - g(y_0)), h)| + M_3|h| \sup_{0\leq t\leq 1} |(0, th)| \\
&\leq M_1M_2|h|(|g(y_0+h) - g(y_0)| + |h|) + M_3|h|^2 \\
&= (M_1M_2(M_1+1) + M_3)|h|^2 \ .
\end{aligned}
$$

Thus, $\lim_{h\to 0} \dfrac{|r(h)|}{|h|} = 0$ and therefore $\hat{f}$ is differentiable. If $y_0 \in B_r(0)$ is a critical point of $\hat{f}$, then

$$f_x(g(y_0), y_0) = 0, \quad f_y(g(y_0), y_0) = 0$$

and being $(x, y) = (0, 0)$ only critical point of f on

$$U = B_\delta(0) \oplus B_r(0),$$

then $y_0 = 0$.

The proof that $(\hat{f})'$ is Lipschitzian on $B_r(0)$, and of items 2., 3. , will be omitted since it is completely elementary. This completes the proof.

□

4. Computation of the local critical groups

Under the hypotheses of the Morse Lemma, Proposition 3.2, if 0 is the only critical point of f and $f(0) = c$. Let V be a closed neighborhood of 0 in H, and a homeomorphism $\psi : V \to \psi(V) \subset H$, $\psi(0) = 0$, $d\psi(0) = I$, such that

$$(f \circ \psi)(x, y) = \frac{1}{2}\langle Ax, x\rangle + \hat{f}(y), \text{ with } g : V \cap N \to N^\perp \text{ and } \hat{f}(y) = f(g(y), y),$$

where $dg(0) = 0$, $g(0) = 0$. Let $C \subset V$ be a closed neighborhood of 0 with $f(0) = \hat{f}(0) = c$. Then the critical groups (over a field F) of V and C satisfy

$$
\begin{aligned}
C_n(f, 0) &= H_n(\{f \leq c\} \cap \psi(C) \,, \{f \leq c\} \cap \psi(C) - \{0\}) \\
&\approx H_n(\{f \circ \psi \leq c\} \cap C \,, \{f \circ \psi \leq c\} \cap C - \{0\}) = C_n(f \circ \psi \,, 0)
\end{aligned}
$$

The Proposition 3.3, allow us to prove the Shifting Theorem, that is due to Gromoll–Meyer [4]; see also K. C. Chang [18] and [19]. The proof that we give here is inspired by treatment of Mawhin–Willem [15, pg. 190].

Now we consider the following conditions:

(i) Let M a Hilbert manifold and $f \in C^{2-}(M, \mathbb{R})$ such the critical points are isolated;

(ii) $X \subset M$ is positively invariant for the flow of gradient field ∇f (i.e. $\sigma(t, u) \in X$ for all $u \in X$ and $0 \leq t < \beta(u)$);

(iii) $a < b$ are real numbers such that $f^{-1}([a,b]) \cap X$ is complete;
(iv) the condition (C) of Palais-Smale over $f^{-1}([a,b]) \cap X$ is satisfied.

Proposition 4.1 (Deformation Lemma). *Under the above conditions, if $f^{-1}(a,b) \cap X$ is free of critical points. Then $\{f \leq a\} \cap X$ is a strong deformation retract of $\{f \leq b\} \cap X - K_b$, where $K_b = \{u \in X : f(u) = b,\ f'(u) = 0\}$; see* [15, pg. 181].

We shall use the following result of Relative Homology:

Lemma 4.1. *Let A be a subset of $\mathbb{R}^p$ containing 0 and let B^k be the $k-$ball. Then, for $k \geq 1$, $H_n(A \times B^k, (A \times B^k) - \{0\}) \approx H_{n-k}(A, A - \{0\})$.*
In particular, if $A = \{0\}$, we obtain

$$\begin{aligned} H_n(B^k, B^k - \{0\}) &\approx H_n(\{0\} \times B^k, (\{0\} \times B^k) - \{0\}) \\ &\approx H_{n-k}(\{0\}, \phi) = \delta_{n-k,\,0} F = \delta_{n,\,k} F. \end{aligned}$$

Now we prove that the critical groups at a critical point depend on the Morse index and the degenerate part.

Proposition 4.2 (Shifting Theorem). *Let $f : U \subset H \to \mathbb{R}$ be a C^1 function defined on a open set and let $0 \in U$ be only critical point of f, $f(0) = c$. Suppose that f is twice differentiable at 0 and that $A = f''(0)$ is a Fredholm operator, with Morse index k finite, so that f' is strongly differentiable at the origin. Then $C_n(f, 0) = C_{n-k}(\hat{f}, 0)$, $n = 0, 1, 2, \cdots$.*

Proof. By Proposition 3.2, we consider the function

$$(f \circ \psi)(x, y) = \frac{1}{2}\langle Ax,\, x\rangle + \hat{f}(y) \ \text{ with } \ f(0) = \hat{f}(0) = c.$$

By Proposition 3.3, $0 \in N = Ker(A)$ is the only critical point of the function $\hat{f} : W \subset N \to \mathbb{R}$, $\hat{f}(y) = f(g(y), y)$, W open set of N, $\hat{f}$ is of class C^1 with $(\hat{f})'$ Lipschitzian on W. Since $dimN$ is finite, $\hat{f}$ satisfy the Palais–Smalle condition over any closed ball. Let $\overline{B}_r(0) \subset W$ be a closed ball.

Now we consider the flow defined by the Cauchy problem

$$\dot{\sigma}(t, y) = -\nabla \hat{f}(\sigma(t, y)),\ \sigma(0, y) = y,\ y \in W.$$

Let $\varepsilon > 0$ be sufficiently small and $V = \overline{B}_{\frac{r}{2}}(0) \cap \{\hat{f} \leq c + \varepsilon\} \subset W$, such that, if $y \in V$, $\sigma(t, y)$ stays in $\overline{B}_r(0)$ for $0 \leq t < \beta(y)$ or $\sigma(t, y)$ stays in $\overline{B}_r(0)$ until $\hat{f}(\sigma(t, y)) \leq c - \varepsilon$ and, therefore the trajectory $\sigma(t, y)$ must cross the level $\hat{f}(y) \equiv c$ at an unique point. Let $X = \overline{Y}$ be the closure in W of the set $Y = \{\sigma(t, y) : y \in V,\ 0 \leq t < \beta(y)\}$. Then, X satisfy the following properties:

(i) X is a neighborhood of 0, closed in W, and X is positively invariant for the flow σ defined by $\dot{\sigma}(t, y) = -\nabla \hat{f}(\sigma(t, y))$, $\sigma(t, y) = y$.
(ii) $0 \in \hat{f}^{-1}([c - \varepsilon, c + \varepsilon]) \cap X$ is the only critical point of $\hat{f}$ and belonging to the interior of $\hat{f}^{-1}([c - \varepsilon, c + \varepsilon]) \cap X$.
(iii) $\hat{f}^{-1}([c - \varepsilon, c + \varepsilon]) \cap X$ is complete, because $\overline{B}_r(0)$ is closed in W, the set $\hat{f}^{-1}([c - \varepsilon, c + \varepsilon]) \cap X$ is contained in $\overline{B}_r(0)$ and closed in $\overline{B}_r(0)$ which is complete. The condition (C) of Palais–Smalle is satisfied over $\hat{f}^{-1}([c - \varepsilon, c + \varepsilon]) \cap X$, because $\hat{f}$ satisfy the condition (C) over $\overline{B}_r(0)$.

Now, we define $X^c = \{\hat{f} \leq c\} \cap X$ and $X^{c+\varepsilon} = \{\hat{f} \leq c+\varepsilon\} \cap X$. We observe that $\hat{f}$ is decreasing during the corresponding deformation σ, obtained of field

$$-\nabla \hat{f} : \ \ \dot{\sigma}(t,y) = -\nabla \hat{f}(t,y), \ \ \sigma(0,y) = y.$$

The set $\hat{f}^{-1}((c,c+\varepsilon]) \cap X = X^{c+\varepsilon} - X^c$ is free of critical points and union of two disjoint subsets S_1 and S_2 such that, for each $y \in S_1$, there is a unique $t(y)$ such that $\hat{f}(\sigma(t(y),y)) = c$ and, for each $y \in S_2$, $\hat{f}(\sigma(t,y)) \to 0$ and , $\sigma(t,y) \to 0$, as $t \to \infty$.

We observe that $\psi(t,y) = \hat{f}(\sigma(t,y))$ satisfies

$$\frac{\partial \psi}{\partial t}(t(y),y)) = -|\nabla \hat{f}(\sigma(t(y),y))|^2 \neq 0$$

and $t(y)$ is continuous by the Implicit Function Theorem. Proposition 4.1 implies X^c is a strong deformation retract of $X^{c+\varepsilon}$.

Let $H = H^- \oplus H^+ \oplus N$ be the orthogonal decomposition into subspaces spanned by the eigenvectors of Fredholm operator $A = f''(0)$ having eigenvalue negative, positive and zero, respectively. A is negative definite on H^-, positive definite on H^+, and $N = KerA$. Let $v = x+y = x^- + x^+ + y$ the corresponding decomposition of any $v \in H$. Define the deformation η of $C = H^- \oplus H^+ \oplus X^{c+\varepsilon}$ by

$$\begin{aligned} \eta : [0,1] \times H^- \times H^+ \times X^{\varepsilon} &\to H^- \times H^+ \times X^{c+\varepsilon} \\ \eta(t,x^-,x^+,y) &= x^- + (1-t)x^+ + \varphi(t,v). \end{aligned}$$

Thus, $H^- \times X^c$ (resp. $(H^- \times X^c) - \{0\}$) is a strong deformation retract of $H^- \times X^{c+\varepsilon}$ (resp. $(H^- \times X^{c+\varepsilon}) - \{0\}$) and we obtain:

$$\begin{aligned} C_n(f,0) &= C_n(f \circ \psi, 0) = H_n(\{f \circ \psi \leq c\} \cap C, \, \{f \circ \psi \leq c\} \cap C - \{0\}) \\ &\approx H_n(H^- \times X^c, \, (H^- \times X^c) - \{0\}). \end{aligned}$$

If $k = dim H^- \geq 0$, by Lemma 4.1, page 9, we obtain:

$$\begin{aligned} C_n(f,0) &\approx H_n(X^c \times \mathbb{R}^k, \, (X^c \times \mathbb{R}^k) - \{0\}) \\ &\approx H_n(X^c \times B^k, \, (X^c \times B^k) - \{0\}) \\ &= H_{n-k}(\{\hat{f} \leq c\} \cap X, \, \{\hat{f} \leq c\} \cap X - \{0\}) \\ &= C_{n-k}(\hat{f}, 0). \end{aligned}$$

This completes the proof.

□

5. Information about dimension of critical groups

It is a known fact in Morse theory: for a function f defined on open set U of a p–dimensional Euclidean space, $f \in C^2(U,\mathbb{R})$, where v is the only critical point, the function f, can be approximated with respect to the C^2 topology by a function $\tilde{f} \in C^2(U,\mathbb{R})$, with critical points in finite number and non degenerate, and that $dim \ C_n(f,v)$ is finite for every n and equal to zero for $n > p$. This fact is proved in the Lemma 8.6 and Theorem 8.5 of book [15] of J. Mawhin and M. Willem where the C^2 case is treated.

Now we consider a function $f : U \to \mathbb{R}$, with low differentiability, defined on open set U of a p–dimensional Euclidean space, $f \in C^1(U,\mathbb{R})$, where v is the only critical point of f, possible degenerate, and f with second derivative at v, and f' strongly differentiable at v. Under these conditions we can affirm that $dim \ C_n(f,v)$ is finite for every n and is zero for $n > p$.

The answer is still true and less assumptions are needed for the function f: If f is a function defined in an open set U of a Euclidean space of dimension p, f is continuously differentiable on U and v is an isolated critical point of f, then $dim\ C_n(f,v)$ is finite for every n and is zero for $n>p$.

This last result is contained in Theorem 3.2 of the paper [3] of C. Li, S. Li, and J. Liu, or in Theorem 1.1 of paper [22] of S. Cingolani and M. Degiovanni, or in Theorem 1.1 of paper [20] of M. Degiovanni (in finite dimension, all assumptions are trivially satisfied). This last result gives also the required information for $n>p$. Since one can consider Alexander-Spanier cohomology, critical groups can be obtained as a limit from open subsets of the Euclidean space of dimension p, hence must vanish for $n>p$.

By Shifting theorem the computation of the critical groups is reduced to a problem in finite dimension, and as consequence we have the following result: Under the assumptions of Proposition 3.2, if 0 is an isolated critical of f and $A=f''(0)$ is a Fredholm operator with finite Morse index k and nullity ν, then the following is true: $dim\ C_n(f,0)$ is finite for every n and equal to zero for $n\notin\{k,\ k+1,\cdots,\ k+\nu\}$.

6. Critical submanifolds of a Hilbert manifold

Let M be a Hilbert manifold. We recall that a connected submanifold K is critical for a function $f\in C^1(M,\mathbb{R})$ if $df(x)=0$, $\forall x\in K$. The tangent space to M at $x\in K$ admits the orthogonal decomposition

$$T_xM=T_xK\oplus H_1(x);\ \ H_1(x)=T_x^{\perp}K.$$

We will suppose f twice differentiable along K and for $x\in K$, $A(x)=\frac{1}{2}d^2f(x)$. If K is critical, $T_xK\subset Ker A(x)$ and since $A(x)$ is symmetric, $A(H_1(x))\subset H_1(x)$.

Definition 6.1. *We will say that K is non-degenerate critical submanifold if $A(x)|H_1(x)$ is an isomorphism for all $x\in K$.*

By the preceding definition, if K is non-degenerate then $T_xK=Ker A(x)$; see [21]. Now, we will suppose that $f\in C^1(M,\mathbb{R})$, twice differentiable along a compact, connected critical submanifold K that can be degenerate, and $A(x)$ depends continuously on $x\in k$.

Definition 6.2. *Under the above hypotheses, if U is a sufficiently small closed neighborhood of K, $f\equiv 0$ in K, the critical groups of K are defined by*

$$C_n(f,K)=H_n(\{f\le 0\}\cap U,\ \{f\le 0\}\cap U-K),\ n=0,\ 1,\ 2,...$$

where F is a field of coefficients.(By excision, the critical groups are independent of U.)

7. The manifold of closed curves

The material covered in this item can be encountered in [23]. We denote by M an $n-$dimensional compact manifold endowed with a Riemannian metric $\langle\,,\rangle$. Let ∇ be the covariant derivative on TM, derived from the Levi-Cività connection. Let S be the parametrized circle $[0,1]/\mathbb{Z}$. We will denote by ΛM and sometimes even simply Λ the set $H^1(S,M)$. Here a map $c:S\to M$ is called H^1, if it is absolutely continuous, and the derivative $\dot c(t)$ (which is defined almost everywhere) is square integrable with respect to the Riemannian metric on M : $\|\dot c(t)\|\in L^2(S)$, *i.e.*, $\int_S\langle\dot c(t),\dot c(t)\rangle_{c(t)}\mathrm{d}t<\infty$. We observe that ΛM has a smooth Riemannian manifold structure (modeled on a Hilbert space), which is associated in a natural way to the Riemannian metric on M. Let $c \in C^\infty(S, M)$ and consider the pull-back diagram:

$$\begin{array}{ccc} c^*TM & \xrightarrow{c^*_\pi} & TM \\ {\scriptstyle \pi^*_c}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{S} & \xrightarrow[c]{} & M \end{array}$$

The Riemannian metric and connection on M can be pulled back to a Riemannian metric and a connection on π^*_c. We denote them by $\langle\,,\rangle_c$ and ∇_c, respectively. Let $\sum(\pi^*_c)$ be the set of all sections of π^*_c and

$$H^0(c^*TM) = L^2(c^*TM) = \{X \in \sum(\pi^*_c) : \|X(t)\|_c \in L^2(S)\}$$
$$H^1(c^*TM) = \{X \in C^0(\pi^*_c) : \nabla_c X \text{ exists a.e. and } \nabla_c X \in H^0(c^*TM)\}.$$

Naturally, $C^k(c^*TM)$ will have the usual meaning for $k = 0, 1, \ldots, \infty$. Then, $H^i(c^*TM)$, $i = 0, 1$, are Hilbert spaces (modulo the relation of being equal a. e.) with respect to the scalar products

$$\langle X\,, Y\rangle_0 = \int_S \langle X(t)\,, Y(t)\rangle_c\,\mathrm{d}t\,, \quad \langle X\,, Y\rangle_1 = \langle X\,, Y\rangle_0 + \langle \nabla_c X\,, \nabla_c Y\rangle_0\,,$$

and we denote by $\|\cdot\|_i$ the relative norms. We also consider

$$C^0(T^*M) = \text{ the set of continuous sections}$$

and endow this vector space with the norm $\|X\|_\infty = \sup\limits_{0\le t\le 1} \|X(t)\|$.

Proposition 7.1. *The inclusions*

$$H^1(c^*TM) \hookrightarrow C^0(c^*TM) \hookrightarrow H^0(c^*TM)$$

are continuous, the first one being compact. More precisely,

$$\|\cdot\|_0 \;\le\; \|\cdot\|_\infty \;\le \sqrt{2}\|\cdot\|_1.$$

For $i = 0, 1$, define $H^i(\Lambda M^*TM) = \bigcup_{c\in\Lambda M} H^i(c^*TM)$.

Proposition 7.2. $p_i : H^i(\Lambda M^*TM) \longrightarrow \Lambda M$, *where* $p_i(X)(t) = \pi(X(t))$, *has the structure of a (Hilbert) vector bundle over* ΛM *and*

$$p_1 : H^1(\Lambda M^*TM) \longrightarrow \Lambda M$$

is isomorphic to $T\Lambda M$.

Without going into details, we will produce an explicit local trivial structure for $H^i(\Lambda M^*TM)$. Let $\pi : TM \to M$ be the smooth vector bundle and

$$K : T(TM) \to TM$$

a connection. Then $T(TM)$ splits into its horizontal and vertical sub-bundle

$$(T^hTM)_x = \ker(K|T_xTM)\;,\; (T^vTM)_y = \ker(\mathrm{d}\pi)_y\;.$$

For $x \in TM, j = 1, 2$, define

$$(\nabla_j \exp)(x) : T_{\pi(x)}M \longrightarrow T_{\exp x}M,$$

by

$$(\nabla_1 \exp)(x) \cdot y = (d\exp)(x) \circ (d\pi|T^hTM)^{-1} \cdot y,$$
$$(\nabla_2 \exp)(x) \cdot y = (d\exp)(x) \circ K(x)^{-1} \cdot y,$$

where

$$K(x) : T^v_xTM \longrightarrow T_{\pi(x)}M$$

is the canonical identification.

Clearly, for $x \in U \subset TM$, $(\nabla_2 \exp)(x)$ is an isomorphism; the maps

$$\widetilde{\phi}^i_c : H^1(U_c) \times H^i(c^*TM) \longrightarrow H^i(\Lambda M^*TM)$$

$$\widetilde{\phi}^i_c(X, Y)(t) = (\nabla_2 \exp)(c^*_\pi X(t)) \cdot (c^*_\pi Y(t))$$

give the required structure, and $(\widetilde{\phi}^1_c)^{-1} \circ (\widetilde{\phi}^1_d)$ is of form

$$(\widetilde{\phi}^{-1}_c \circ \widetilde{\phi}_d \, , \, d(\widetilde{\phi}^{-1}_c \circ \widetilde{\phi}_d)),$$

so the last assertion follows.

For any $c \in \Lambda M$, $\dot{c}(t) \in H^o(\Lambda M^*TM)$. This gives a section

$$\partial : \Lambda M \to H^o(\Lambda M^*TM).$$

Proposition 7.3. *∂ is a smooth map.*

Theorem 7.1. *The bundle $p_i : H^i(\Lambda M^*TM) \longrightarrow \Lambda M$ has a (unique) Riemannian metric characterized by the following property: For $c \in C^\infty(S, M)$ the metric on $p_i^{-1}(c) = H^i(c^*TM)$ is given by the scalar product $\langle \, , \rangle_i$.*

Naturally we will keep denoting this Riemannian metric by $\langle \, , \rangle_i$. In particular,

$$T\Lambda M \cong H^1(\Lambda M^*TM)$$

has a natural Riemannian structure that we will denote also by $\langle \, , \rangle_i$.

7.1. **The integral energy of a Finsler manifold.** We consider the manifold $\Lambda M = H^1(S, M)$ of H^1-maps of the circle S into M. Let M be a compact manifold and $F : TM \to R$ a Finsler metric on M (see [14], chapter 1, for details). The function $L = F^2 : TM \to \mathbb{R}$ induces a map $E : \Lambda M \to \mathbb{R}$ by $E(c) = \int_S L(\dot{c}(t))\mathrm{d}t$ called energy integral or simply the energy. Let $c \in C^\infty(S, M)$ and $(\phi_c, H^1(U_c))$ a coordinate system near c and $E_c = E \circ \phi_c$, where $U_c = ({c^*_\pi}^{-1})U$, and U is an open set containing the zero section in TM. Then E_c is the composite of the following maps:

$$H^1(U_c) \xrightarrow{I \times \partial_c} H^1(U_c) \times H^0(c^*TM) \xrightarrow{\widetilde{\lambda}_c} L^1(S) \longrightarrow \mathbb{R}$$

where the last map is just integration and $\widetilde{\lambda}_c$ is induced by the fiber map

$$\lambda_c : U_c \oplus c^*TM \to S \times \mathbb{R},$$

$$\lambda_c(x, y) = (\pi^*_c x, L((\nabla_2 \exp)(c^*_\pi x) \cdot c^*_\pi y)), \text{where } L = F^2.$$

We note that $\widetilde{\lambda}_c$ is well-defined on all of $H^1(U_c) \times H^0(c^*TM)$. In fact, for $(X, Y) \in H^1(U_c) \times H^0(c^*M)$,

$$\int_S L((\nabla_2 \exp)(c^*_\pi X(t)) \cdot c^*_\pi Y(t))\mathrm{d}t \;\leq\; K \int_S \|(\nabla_2 \exp)(c^*_\pi X(t))\|^2 \|Y(t)\|^2 \mathrm{d}t$$

which is bounded since $\|X(t)\|_\infty$ is small and $Y(t) \in H^0(c^*TM) = L^2(c^*TM)$.

For any $t \in S$, consider the restriction of λ_c to the fiber

$$\lambda_t : (U_c)_t \oplus (c^*TM)_t \to \mathbb{R}.$$

If we denote by x, y the variables in the first and second factor, respectively, we have:

(1) λ_t and $\dfrac{\partial \lambda_t}{\partial x}$ are positively homogeneous of degree 2 in y,

(2) $\dfrac{\partial \lambda_t}{\partial y}$, $\dfrac{\partial^2 \lambda_t}{\partial x \partial y}$, $\dfrac{\partial^2 \lambda_t}{\partial y \partial x}$ are positively homogeneous of degree 1 in y,

(3) $\dfrac{\partial^2 \lambda_t}{\partial y^2}$ is positively homogeneous of degree zero in y.

The closed geodesics problem for the Finsler metric can be posed in an analogous manner to the one for the Riemannian metrics, and the critical points for the function

$$E : \Lambda M \to \mathbb{R}\,, \quad E(c) = \int_S F^2(\dot{c}(t))\mathrm{d}t$$

are exactly the closed geodesics (see [9]).

The gradient field of E is defined by

$$\langle \nabla E(c)\,,\, X \rangle_1 = dE(c).X \text{ for all } X \in T_c\Lambda M.$$

The energy function $E : \Lambda M \to \mathbb{R}$ possesses many properties that are necessary for the development of the theory of Morse of $(\Lambda M, E)$. Among these properties for the function $E : \Lambda M \to \mathbb{R}$ we have the following two:

i) E is C^{2-} (i.e. it is C^1 and the differential is locally Lipschitzian) and therefore E is strongly differentiable.

ii) If M is compact, E satisfy the condition (C) of Palais and Smale:
"Let (c_n) be a sequence on ΛM such that the sequence $(E(c_n))$ is bounded and $(\|\nabla E(c_n)\|_1)$ tends to zero. Then (c_n) has limit point and any limit point is a critical point of E".

Remark 7.1. *Condition (C) should be viewed as a substitute for the fact that ΛM is not locally compact.*

The proof that E is differentiable with differential locally Lipschitzian and that E satisfies the condition (C) of Palais–Smale is due to F. Mercuri and can be found in the paper [9]. To prove $i)$ it is sufficient to show that $\widetilde{\lambda}_c$ is C^{2-} with

$$(d\widetilde{\lambda}_c)(X,Y)(t) \;=\; (d_f\lambda_c)(X(t),Y(t)), \;\; (X,Y) \in H^1(U_c) \times H^0(c^*TM)$$

where d_f denotes derivative on the fiber and in this case

$$(d_f\lambda_c)(X(t),Y(t))(X_1(t),Y_1(t)) = \frac{\partial \lambda_t}{\partial x}(X(t),Y(t)) \cdot X_1(t) + \frac{\partial \lambda_t}{\partial y}(X(t),Y(t)) \cdot Y_1(t)$$

with $(X_1, Y_1) \in H^1(U_c) \times H^0(c^*TM)$. The other property that is necessary for the Morse theory of Finsler energy function is the following:

iii) The derivative of the Finsler energy function $E : \Lambda M \to \mathbb{R}$ is strongly differentiable on the regular curves and particularly on the closed geodesics.

Now our main purpose of this paper is to prove the property *iii*) enunciated above.

Lemma 7.1. *Homogeneity lemma: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuous, C^∞ on $\mathbb{R}^n - \{0\}$, and positively homogeneous of degree α. Then for all $x, y \in \mathbb{R}^n$:*

(a) *If $\alpha = 1$, there is a constant K with $\|f(x) - f(y)\| \leq K\|x - y\|$.*
(b) *If $\alpha = 2$, there are constants K_1, K_2 with*

$$\|f(x) - f(y)\| \leq K_1\|x - y\|^2 + K_2\|x - y\|\|y\|.$$

Also, the other statement in (b) is

$$\|f(x) - f(y)\| \leq K\,\|x - y\|.max(\|x\|, \|y\|),\ \textit{where}\ K = max(\|4K_1\|, \|4K_2\|).$$

(c) *If $\alpha = 0$, then f is bounded.*

The proof of Lemma 7.1 will be omitted since it is completely elementary.

8. Strong differentiability of derivative of the Finsler energy in a critical sub-manifold

If we compute the second derivative of $E : \Lambda M \to \mathbb{R}$ we see that we need to use the second derivative of F^2 so that we can carry out the computation only at regular curves, in particular geodesics. The critical points of function $E : \Lambda M \to \mathbb{R}$ are exactly the closed geodesics. Now if c is a closed geodesic the orbit of c under the natural action of $SO(2)$ will give a critical sub-manifold.

Theorem 8.1. *The derivative of the Finsler energy function $E : \Lambda M \to \mathbb{R}$ is strongly differentiable on the regular curves and particularly on the closed geodesics.*

Proof. Let's now prove that the expression

$$\begin{aligned}
[d^2\widetilde{\lambda_c}(A, B)](X_1, Y_1)(X_2, Y_2) &= \frac{\partial^2\lambda_t}{\partial x^2}(A, B)X_1X_2\ + \frac{\partial^2\lambda_t}{\partial x\partial y}(A, B)Y_1X_2 + \\
&+ \frac{\partial^2\lambda_t}{\partial y\partial x}(A, B)X_1Y_2\ + \frac{\partial^2\lambda_t}{\partial y^2}(A, B)Y_1Y_2 \\
&= d_f^2\lambda_c(A, B)(X_1, Y_1)(X_2, Y_2),
\end{aligned}$$

defines the second derivative of $\widetilde{\lambda_c}$. Then, for some $s \in [0, 1]$ and

$$(X_1, Y_1) \in H^1(U_c) \times H^0(c^*TM),$$

small enough, we consider the remainder $R(X - W, Y - Z)$ of definition of strongly differentiable function:

$$\|R(X-W,Y-Z)\|_{L_1} =$$
$$= \left\| \left[d_f\lambda_c(X,Y)\right](X_1,Y_1) - \left[d_f\lambda_c(W,Z)\right](X_1,Y_1) - \left[d_f^2\lambda_c(A,B)\right](X-W,Y-Z)(X_1,Y_1) \right\|_{L_1}$$

$$\begin{aligned}=&\|d_f^2\lambda_c(X+s(X-W),Y+s(Y-Z))(X-W,Y-Z)(X_1,Y_1)-\\ &-d_f^2\lambda_c(A,B)(X-W,Y-Z)(X_1,Y_1)\|_{L_1}\leq\end{aligned}$$

$$\begin{aligned}\leq&\int_S\|\frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial x^2}(A,B)\|\cdot\|X-W\|\cdot\|X_1\|\mathrm{d}t+\\ &+\int_S\|\frac{\partial^2\lambda_t}{\partial x\partial y}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial x\partial y}(A,B)\|\cdot\|Y-Z\|\cdot\|X_1\|\mathrm{d}t+\\ &+\int_S\|\frac{\partial^2\lambda_t}{\partial y\partial x}(X+s(X-W),Y+s(Y-Z))-\frac{\partial\lambda_t}{\partial y\partial x}(A,B)\|\cdot\|X-W\|\cdot\|Y_1\|\mathrm{d}t+\\ &+\int_S\|\frac{\partial^2\lambda_t}{\partial y^2}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial y^2}(A,B)\|\cdot\|Y-Z\|\cdot\|Y_1\|\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\leq&\sqrt{2}\|X-W\|_1\cdot\|X_1\|_\infty\{\int_S\|\frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial x^2}(A,B)\|^2\mathrm{d}t\}^{\frac{1}{2}}+\\ &+\|Y-Z\|_0\cdot\|X_1\|_\infty\{\int_S\|\frac{\partial^2\lambda_t}{\partial x\partial y}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial x\partial y}(A,B)\|^2\mathrm{d}t\}^{\frac{1}{2}}+\\ &+\sqrt{2}\|X-W\|_1\cdot\|Y_1\|_\infty\{\int_S\|\frac{\partial^2\lambda_t}{\partial y\partial x}(X+s(X-W),Y+s(Y-Z))-\\ &-\frac{\partial^2\lambda_t}{\partial y\partial x}(A,B)\|^2\mathrm{d}t\}^{\frac{1}{2}}+M\|Y_1\|_\infty\cdot\|Y-Z\|_0\end{aligned}$$

$$\begin{aligned}\leq&\sqrt{2}\cdot\|Y_1\|_\infty\cdot\|(X-W,Y-Z)\|_{H^1\times H^0}\\ &\left[\int_S\|\frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial x^2}(A,B)\|^2\mathrm{d}t\right]^{\frac{1}{2}}\\ +&\|X_1\|_\infty\cdot\|(X-W,Y-Z)\|_{H^1\times H^0}\\ &\left[\int_S\|\frac{\partial^2\lambda_t}{\partial x\partial y}(X+s(X-W),Y+s(Y-Z))-\frac{\partial^2\lambda_t}{\partial x\partial y}(A,B)\|^2\mathrm{d}t\right]^{\frac{1}{2}}\\ +&\sqrt{2}\|Y_1\|_\infty\cdot\|(X-W,Y-Z)\|_{H^1\times H^0}\{\int_S\|\frac{\partial^2\lambda_t}{\partial y\partial x}(X+s(X-W),Y+s(Y-Z))-\\ -&\frac{\partial^2\lambda_t}{\partial y\partial x}(A,B)\|^2\mathrm{d}t\}^{\frac{1}{2}}+M\|Y_1\|_\infty\cdot\|(X-W,Y-Z)\|_{H^1\times H^0}.\end{aligned}$$

The proof that the derivative of the Finsler energy function $E : \Lambda M \to \mathbb{R}$ is strongly differentiable on the regular curves and particularly on the closed geodesics, will be concluded if we show that the expressions

$$\int_S\left\|\frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),Y+s(Y-Z))\;-\;\frac{\partial^2\lambda_t}{\partial x^2}(A,B)\right\|^2\mathrm{d}t$$

and

$$\int_S\left\|\frac{\partial^2\lambda_t}{\partial x\partial y}\left(X+s(X-W),Y+s(Y-Z)\right)-\frac{\partial^2\lambda_t}{\partial x\partial y}(A,B)\right\|^2\mathrm{d}t$$

have limits equal to zero when $(X,Y)\to(A,B)$ and $(W,Z)\to(A,B)$.

On one hand,

$$\int_S \left\| \frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),Y+s(Y-Z)) \,-\, \frac{\partial^2\lambda_t}{\partial x^2}(A,B) \right\|^2 \mathrm{d}t \le$$

$$\le \int_S [\| \frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),Y+s(Y-Z)) \,-\, \frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),B)\| +$$

$$+ \| \frac{\partial^2\lambda_t}{\partial x^2}(X+s(X-W),B) \,-\, \frac{\partial^2\lambda_t}{\partial x^2}(A,B)\|]^2 \mathrm{d}t$$

$$\le \int_S [K_1 \cdot (\|Y-B\| + \|Y-Z\|)^2 + K_2\|B\| \cdot (\|Y-B\| + \|Y-Z\|) +$$

$$+ K_3(\|X-A\| + \|X-W\|)]^2 \mathrm{d}t.$$

On the other hand,

$$\int_S \left\| \frac{\partial^2\lambda_t}{\partial x \partial y}(X+s(X-W),Y+s(Y-Z)) \,-\, \frac{\partial^2\lambda_t}{\partial x \partial y}(A,B) \right\|^2 \mathrm{d}t \le$$

$$\le \int_S [\| \frac{\partial^2\lambda_t}{\partial x \partial y}(X+s(X-W),Y+s(Y-Z)) \,-\, \frac{\partial^2\lambda_t}{\partial x \partial y}(X+s(X-W),B)\| +$$

$$+ \| \frac{\partial^2\lambda_t}{\partial x \partial y}(X+s(X-W),B)\| \,-\, \frac{\partial^2\lambda_t}{\partial x \partial y}(A,B)\|]^2 \mathrm{d}t$$

$$\le \int_S [K_4\|Y-B\| + \|Y-Z\|) + K_5(\|X-A\| + \|X-W\|)]^2 \mathrm{d}t$$

□

9. Critical submanifolds of the Hilbert manifold ΛM for Finsler energy E

Let M an n-dimensional compact manifold endowed with a Riemannian metric $\langle\, ,\rangle$. Let $c \in \Lambda M$ be a closed geodesic of a Finsler metric $F : TM \to \mathbb{R}$ of multiplicity $m \ge 1$, i.e. $c(t) = c(t + \frac{1}{m})$, for all $t \in [0,1]$. From the index form $d^2E(c)$ associated to the energy $E : \Lambda M \to \mathbb{R}$, $E(c) = \int_S F^2(\dot{c}(t))\mathrm{d}t$, we obtain in the usual way a self-adjoint operator $A_c :\ T_c\Lambda M \to T_c\Lambda M$ defined by

$$d^2E(c)(X,Y) = \langle A_cX, Y\rangle_1 = \langle X, A_cY\rangle_1$$

and A_c is of the form $A_c = I + K_c$, where I is the identity and K_c is a compact operator in the space $(T_c\Lambda M, \|\cdot\|_1)$. Therefore A_c is a Fredholm operator.

Let

$$T_c\Lambda M = T_c^+\Lambda M \oplus T_c^-\Lambda M \oplus T_c^0\Lambda M$$

be the orthogonal decomposition of tangent space $T_c\Lambda M$ into sub-spaces spanned by the eigenvectors of A_c having eigenvalue > 0, < 0 and $= 0$ respectively. Theses sub-spaces are $A-invariant$ and A is negative definite on $T_c^-\Lambda M$ and positive definite on $T_c^+\Lambda M$. Then $dim\, T_c^-\Lambda M$ and $dim\, T_c^0\Lambda M$ are finite. We call $dim\, T_c^-\Lambda M$ and $dim\, T_c^0\Lambda M - 1$ the index and the nullity of c, respectively. Denote by $T_c'\Lambda M$ the sub-space of $T_c\Lambda M$ of codimension 1 which is orthogonal to $\dot{c} \in T_c\Lambda M$. From the above decomposition we get the orthogonal decomposition

$$T_c'\Lambda M = T_c^+\Lambda M \oplus T_c^-\Lambda M \oplus T_c'^{\,0}\Lambda M,$$

where $T_c'^0\Lambda M = T_c^0\Lambda M \cap T_c'\Lambda M$ consists of the periodic Jacobi fields along c which are orthogonal to $\dot{c}$ and $index(c) = dim\, T_c'\Lambda M$, $nullity(c) = dim\, T_c'^0\Lambda M$. We do not assume that $nullity\,(c) = 0$, i.e. that c is non-degenerate.

The closed orbit $S\cdot c$ of a closed geodesic c, is a closed submanifold of ΛM critical for energy $E : \Lambda M \to R$, under the S-action:

$$S \times \Lambda M \to \Lambda M, \ \ (z, d) \mapsto z \cdot d$$

with

$$z \cdot d(t) = d(t + r), \text{where } z = e^{2\pi ir} \in S, \ 0 \leq r \leq 1.$$

The critical sub-manifold $S.c$ is compact and connected. The connectedness of $S.c$ implies that the index and nullity of E along $S.c$ are well-defined, i.e. $T^-_{z.c}\Lambda M$ and $T'^0_{z.c}\Lambda M$ has constant dimensions (see [21] and [23]).

The closed geodesic c is non-degenerate, if and only if, the orbit $S \cdot c$ is non-degenerate. This is equivalent to saying that the $nullity(c) = 0$ or the

$$nullity(d^2E(c)) = dim\, T_c^0\Lambda M = 1.$$

Let $\mu = \mu(S \cdot c) : N \to S$ be the normal bundle of closed geodesic c over S, induced for embedding $z \in S \to z^{\frac{1}{m}} \cdot c \in \Lambda M$, $z^{\frac{1}{m}} \cdot c(t) = e^{2\pi i \frac{r}{m}} \cdot c(t) == c(t + \frac{r}{m})$ where $m =$multiplicity of c and $0 \leq r \leq 1$. Note that here we do not assume that $S \cdot c$ is a non-degenerate critical sub-manifold.

Let $\mu = \mu^+ \oplus \mu^- \oplus \mu^0$ be the splitting of the normal bundle, determined by the splitting

$$T_c'\Lambda M = T_c^+\Lambda M \oplus T_c^-\Lambda M \oplus T_c'^0\Lambda M$$

of the fiber.

Using the exponential map, exp, of Levi-Cività connection, we can identify the total space $D = D(S \cdot c)$ of a sufficiently small $\varepsilon - disc$ bundle $D_\varepsilon\mu$ of μ with a open neighborhood of $S \subset N$. The tangent bundle $T\Lambda M$ has in the usual way a Riemannian metric defined by

$$\langle X, Y\rangle_1 = \langle X, Y\rangle_0 + \langle \nabla_{\dot{c}}X, \nabla_{\dot{c}}Y\rangle_0.$$

We use the exponential map to pull the Riemannian metric and Finsler energy integral E back onto D. The action of S on D respects these quantities. For $z \in S$, we denote by D_z the fiber over z in $D\mu$. The restriction of E to D_z will be denoted by E_z. Actually, for our purposes a different metric on D is useful.

Let m be the multiplicity of c. Define on D the following modification of the Riemannian metric

$$\langle X, Y\rangle_m = m^2\langle X, Y\rangle_0 + \langle \nabla_{\dot{\sigma}}X, \nabla_{\dot{\sigma}}Y\rangle_0, \, X, Y \in T_\sigma\Lambda M.$$

On the tangent space of each $X \in D$, the metric $\langle\, ,\rangle_m$ is clearly equivalent to the metric $\langle\, ,\rangle_1$. The index and the nullity of c are not affected by this change of the metric. The function E_z is C^1 and E_z is twice differentiable at O_z with $\mathrm{d}E_z$ strongly differentiable at the origin O_z and $\mathrm{d}E_z(O_z) = 0$.

Using the previously concepts, we now obtain the generalized Morse lemma for degenerate critical points with low differentiability for $E_z = E|D_z$.

Proposition 9.1. *(Generalized Morse Lemma for $E_z = E|D_z$).*
Let c be a closed geodesic of a Finsler metric $F : TM \to \mathbb{R}$, of nullity $l \geq 0$ and

multiplicity $m \geq 1$. We put $\mu = \mu^+ \oplus \mu^- \oplus \mu^0$ the splitting of normal bundle, determined by the splitting of the fiber $T_c^{'}\Lambda M$:

$$T_c^{\prime}\Lambda M = T_c^{+}\Lambda M \oplus T_c^{-}\Lambda M \oplus T_c^{\prime 0}\Lambda M$$

where fiber dimension of μ^0 is l. We put $\mu^* = \mu^+ \oplus \mu^-$, i.e., $\mu = \mu^* \oplus \mu^0$. Denote by O_+,O_-, O_0 and O_* the zero sections of bundles μ^+, μ^-, μ^0 and μ^*, respectively. Then,

(1) There is a local homeomorphism ψ of D, such that $\psi : V \to \psi(V) \subset D$, where V is an open set, $0 \in V$, $\psi(0) = 0$, with ψ differentiable in 0 and $d\psi(0) = I$.

(2) There is a continuous function $g : D_0 \to D_* = D_+ \oplus D_-$ that is strongly differentiable in $O_z \in D_0$, and $g(0_z) = 0_z$, $\mathrm{d}g(O_z) = 0$.

(3) There is a section $z \in S \to P_z \in L(\mu^*, \mu^*)$ with

$$P_z : X \in D_+ \oplus D_- \to P_z(X) \in D_+$$

being an orthogonal bundle projection such that, for $(X, Y) \in D_* \oplus D_0$

$$\begin{aligned} E_z(X,Y) &\equiv (E_z \circ \psi)(X,Y) = d_X^2 E_z(O_z)(X,X) + E_z(g(Y),Y) \\ &= \langle A_z(0,0)X, X\rangle_m + E_z(g(Y),Y) \\ &= \|P_z(X)\|_m^2 - \|(I - P_z)(X)\|_m^2 + E_z(g(Y),Y) \end{aligned}$$

where $\|X\|_m^2 = \langle A_z(0,0)X^+, X^+\rangle_m - \langle A_z(0,0)X^-, X^-\rangle_m$.
The homeomorphism

$$\psi|D_0 : D_0 \to \psi(D_0) \subset D$$
$$(\psi|D_0)(Y) = g(Y) + Y,\ d(\psi|D_0)(0) = I_N,\ N = T_c^{\prime 0}\Lambda M,$$

define a topological sub-manifold $\psi(D_0) \subset D$, called characteristic sub-manifold at c.

10. Homological invariants of the energy E at the isolated critical submanifold

Now we will define homological invariants of a closed F- geodesic c of multiplicity $m \geq 1$ and homological invariants of isolated critical orbits $S \cdot c$ of the energy E on ΛM. Let c a closed geodesic of multiplicity m, and $c_0 \in \Lambda M$ a prime closed geodesic defined by $c_0(t) = c(\frac{t}{m})$. Put $E(c) = m^2 E(c_0) = k_m$. Let $\mu = \mu(S \cdot c)$ be the normal bundle over S induced from

$$z \in S \mapsto z^{\frac{1}{m}} \cdot c \in \Lambda M$$

and let

$$\mu(S \cdot c) = \mu^+(S \cdot c) \oplus \mu^-(S \cdot c) \oplus \mu^0(S \cdot c)$$

be the splitting according to the sign of the eigenvalues, introduced earlier.

We denote by $E_z(X, Y)$ the local representation of $E_z = E|D_z$ given by the generalized Morse lemma:

$$E_z : (D_z(S \cdot c), O_z(S \cdot c)) \to (\mathbb{R}, k_m)$$

$$E_z(X,Y) \equiv (E_z \circ \psi)(X,Y) = \|P_z(X)\|^2 - \|(I - P_z)(X)\|^2 + E_z(g(Y),Y)$$

and by $E_{0,z}$ the function given by

$$\widehat{E}_z : (D_{0,\,z}(S \cdot c), O_{0,\,z}(S \cdot c)) \to (\mathbb{R}, k_m)\,,\ \ \widehat{E}_z(Y) = E_z(g(Y),Y)$$

where $k_m = E(c) = m^2 E(c_0)$, $O_z(S \cdot c)$ denote the origin of fiber $D_z(S \cdot c)$ and $O_{0,z}(S \cdot c)$ is the origin of $D_{0,z}(S \cdot c)$.

The homology groups (over the field of rational numbers) defined by

$$C_i(E,c) \,=\, H_i([B_{z,\,\varepsilon}(c)\cap\{E_z\leq k_m\}],\, B_{z,\,\varepsilon}(c)\cap\{E_z\leq k_m\}-\{0\})$$

$$C_i(\widehat{E}_z,c) \,=\, H_i([D^0_{z,\,\varepsilon}(c)\cap\{\widehat{E}_z\leq k_m\}],\, D^0_{z,\,\varepsilon}(c)\cap\{\widehat{E}_z\leq k_m\}-\{0\})$$

are the homological invariants associated to the closed F-geodesic c, of multiplicity m, and $C_i(E,c)$ is the characteristic invariant, $B_{z,\,\varepsilon}(c)$ is an open disc with center at the origin of fiber D_z and radius $\varepsilon>0$, sufficiently small, $D^0_{z,\,\varepsilon}(c)$ is a small open disc in $(D_0)_z$ of same center that $(D_0)_z$. By excision, these critical groups are independent of $B_{z,\,\varepsilon}(c)$ and $D^0_{z,\,\varepsilon}(c)$.

The numbers $b_i(c)=dim\, C_i(E,c)$ are called the i-th type number, and $b^0_i(c)= dim\, C_i(\widehat{E}_z,c)$ are the i-th singular type number of closed F-geodesic c. Since all constructions are made equivariantly with respect to the S-action on $D(S\cdot c)$, the homology groups and the type numbers are independent from the choice of $z\in S$. These homological invariants are independent the choice of the metric. Let $\langle\,,\rangle$ and $\widetilde{\langle\,,\rangle}$ be two Riemannian metrics in the compact manifold M, then are verified the following conditions:

(1) The unitary tangent bundles of M, in the metrics $\langle\,,\rangle$ and $\widetilde{\langle\,,\rangle}$ are compact. Therefore the induced norms $\|\,,\|$ and $\widetilde{\|\,,\|}$ in each tangent space T_xM are equivalents.

(2) A map $c:S\to M$ is H^1 in the metric $\langle\,,\rangle$, if and only if, c is H^1 in the metric $\widetilde{\langle\,,\rangle}$. Therefore the manifold ΛM and the vector spaces $H^0(c^*TM)$ and $T_c\Lambda M \,=\, H^1(c^*TM)$ do not depend on the used metric for defining them.

(3) In the space $T_c\Lambda M \,=\, H^1(c^*TM)$ with the scalar products

$$\langle X,Y\rangle_1 \,=\, \langle X(t),Y(t)\rangle_0+\langle\nabla_c X(t),\nabla_c Y(t)\rangle_0$$

$$\widetilde{\langle X,Y\rangle}_1 \,=\, \widetilde{\langle X(t),Y(t)\rangle}_0+\widetilde{\langle\widetilde{\nabla}_c X(t),\widetilde{\nabla}_c Y(t)\rangle}_0$$

the induced norms $\|\,,\|_1$ and $\widetilde{\|\,,\|}_1$ are equivalents. Therefore, the subspace $T'_c\Lambda M$ of $T_c\Lambda M$ of codimension 1 which is orthogonal to $\dot{c}\in T_c\Lambda M$, and the subspaces $T^+_c\Lambda M$ and $T^-_c\Lambda M$ where $d^2E(c)$ is positive definite and negative definite, do not depend on the Riemannian metric of M.

(4)The homological invariants associated to a closed $F-$geodesic are independent of the choice of the metric, but $D_z,\ E_z$ depend on the metric.
Remark: The proof of above properties will be omitted since it is elementary.

An immediate consequence of Proposition 4.2, page 9, is the following proposition.

Proposition 10.1 (Shifting Theorem for Finsler energy)**.** *The characteristic invariant $C_i(E,c)$ of a closed F-geodesic c, together with the index $\lambda=dim\, T^-_c\Lambda M$, and nullity $l\geq 0$ determines $C_i(E,c)$ completely by*

$$C_i(E,c) \,=\, C_{i-\lambda}(\widehat{E}_z,c).$$

By Proposition 4.2 (Shifting theorem) the computation of the critical groups is reduced to a problem in finite dimension, and as consequence we have the following result: $dim\ C_i(E, c)$ is finite for every i and equal to zero for $i \notin \{\lambda,\ \lambda+1, \cdots,\ \lambda+l\}$ (see, [3], [20], [22]), see also papers on the Morse theory [4], [5], [18], [19].

Now will define a local homological invariant $C_i(E, S\cdot c)$ of the energy E at the isolated critical orbit $S.c$ by

$$C_i(E, S\cdot c) \,=\, H_i(B_\varepsilon(S\cdot c) \cap \{E(S\cdot c) \leq k\}\,,\, B_\varepsilon(S\cdot c) \cap \{E(S\cdot c) \leq k\} - O(S\cdot c))$$

where $B_\varepsilon(S\cdot c) \,=\, \bigcup_{z\in S} B_{z,\varepsilon}(S\cdot c)$ is a small tubular neighborhood, which is a normal bundle of small open discs.

The i-th type number of an isolated critical orbit $S\cdot c$ of energy E is defined by $b_i(S\cdot c) \,=\, dim\, C_i(E, S\cdot c)$. The type number $b_i(S\cdot c)$ of a critical orbit $S\cdot c$ and the singular type number of closed F-geodesic c satisfy the inequality

$$b_i(S\cdot c) \,\leq\, 2[b^\circ_{i-\lambda}(c) \,+\, b^\circ_{i-\lambda-1}(c)]$$

that is obtained making use of the homology theory of action of finite groups and all homological invariants are taken with respect to coefficients in a field of characteristic zero, which is necessary when we use the transfer map as is done here (see [5] and [11]). The inspection of the inequalities above is as follows: We denote by W_c and W_c^- the sets

$$W_c \,=\, [B_\varepsilon(c) \cap \{E \leq k\}]\,, \quad W_c^- \,=\, B_\varepsilon(c) \cap \{E \leq k\} - \{c\}$$

where $E(c) \,=\, k$ and by W and $W^-$the sets $S\cdot W_c$ and $S\cdot W_c^-$ respectively. Now we can write the pair

$$(W, W^-) \,=\, (S\times W_c,\, S\times W_c^-)/\Gamma \,=\, ((S\times W_c)/\Gamma,\, (S\times W_c^-)/\Gamma),$$

where the isotropy group Γ acts on trivial bundle $S\times W_c$ by covering transformations.

Hence,

$$H_i(W, W^-) \,=\, H_i((S\times W_c)/\Gamma,\, (S\times W_c^-)/\Gamma)$$

is isomorphic to the sub-space

$$H_i(S\times W_c,\, S\times W_c^-)^\Gamma,$$

of all elements in $H_i(S\times W_c, S\times W_c^-)$ which are kept fixed under the induced operation of Γ on the homology. Observing that Γ acts trivially on $H_i(S)$ we obtain

$$C_i(E, S\cdot c) \,=\, H_i(S) \otimes H_i(W_c,\, W_c^-)^\Gamma \subset H_i(S) \otimes C_i(E, c).$$

The invariant $C_i(E, S\cdot c)$ is of finite type as $C_i(E, c)$, i.e., C_i is finite dimensional and $C_i \,=\, 0$ for almost all i. The fact that $C_{i+\lambda}(E, c) \,=\, C_i(\widehat{E}, c)$ where $\lambda \,=\, index\,(c)$ the last equality above yields

$$C_i(E, S\cdot c) \subset V_i \oplus V_i\,, \quad V_i \,=\, C_{i-\lambda}(\widehat{E}, c) \,\oplus\, C_{i-\lambda-1}(\widehat{E}, c)$$

that in terms of numerical invariants $b_i(S\cdot c) \,=\, dim\, C_i(E, S\cdot c)$ and $b^\circ_i(c) \,=\, dim\, C_i(\widehat{E}, c)$ reads as

$$b_i(S\cdot c) \,\leq\, 2[b^\circ_{i-\lambda}(c) \,+\, b^\circ_{i-\lambda-1}(c)].$$

Let c be a closed F-geodesic and $S.c$ the associated critical sub-manifold. If those critical sub-manifolds are non-degenerate (which is a generic condition on the space of Finsler metrics, see [24]), in this case we compute the index of E and prove that index of the non-degenerate critical sub-manifold obtained by covering m times a given closed F-geodesic and its rotation, is a multiple of the original one.

At this point it is not difficult to obtain the following analogue of the Gromoll-Meyer theorem, without the non-degeneracy hypothesis:

Theorem 10.1. (*Gromoll-Meyer*): *Let (M, F) be a n-dimensional compact simply-connected Finsler manifold satisfying:"If b_k denotes the k-th rational Betti number of ΛM, there is a sequence $k_n \to \infty$ such that $b_{k_n} \to \infty$". Then M has infinitely many closed F-geodesics (geometrically distinct).*

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