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SYMMETRIC TENSOR RANK AND THE IDENTIFICATION OF A POINT USING LINEAR SPANS OF AN EMBEDDED VARIETY

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Abstract. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Fix $P \in \mathbb{P}^n$. In this paper we discuss the minimal integer $\sum_{i=1}^{k} \sharp(S_i)$ such that $S_i \subset X$ and $\{P\} = \cap_{i=1}^{k}\langle S_i\rangle$, where $\langle\ \rangle$ denote the linear span (in positive characteristic sometimes this integer is $+\infty$). We use tools introduced for the study of the X-rank of P. Our main results are when X is a Veronese embedding of $\mathbb{P}^m$ (it is related to the symmetric tensor rank of P) or when X is a curve.

1. Introduction

Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S\rangle$, where $\langle\ \rangle$ denote the linear span. Let $ir_X(P)$ be the minimal integer s such that there are finite sets $S_i \subset X$, $i \geq 1$, such that $\sharp(S_i) \leq s$ for all i and $\{P\} = \cap_{i\geq 1}\langle S_i\rangle$. We prove that $ir_X(P) < +\infty$ if $\mathrm{char}(\mathbb{K}) = 0$ (Proposition 3), but we show that in positive characteristic this is not true in a few cases (Proposition 3). We call $ir_X(P)$ the *identification rank* of P with respect to X or the X*-identification rank* of P. Let $\alpha(X,P)$ be the minimal integer x such that there are finitely many finite sets $S_i \subset X$, say $S_1,\dots,S_k$, such that $\{P\} = \cap_{i=1}^{k}\langle S_i\rangle$ and $\sum_{i=1}^{k}\sharp(S_i) = x$ (we don't fix the integer k and we don't assume that the sets S_i are disjoint, although the last condition is always satisfied if $k = 2$). The integer $\alpha(X,P)$ is the minimal number of points of X needed to identify P among all the points of $\mathbb{P}^n$ using only the operations of linear algebra: first taking several linear spans of points of X and then taking the intersection of these linear subspaces. It is the analogous in projective geometry of the minimal number of photos needed to identify a point of $\mathbb{R}^3$. With a smaller number of points we may only identify a linear subspace, L, containing P, but we cannot distinguish P from the other points of $\mathbb{P}^n$. One could allow both intersections and unions of

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linear spaces $\langle S_i\rangle$, $S_i \subset X$, but obviously in this way the minimal number $\sum_i \sharp(S_i)$ is at least the integer $\alpha(X,P)$ as we defined it. We say that $\alpha(X,P)$ is the *identification number* of P with respect to X. This concept has an obvious geometric meaning, but as in the case of the usual X-rank other related technical definitions may help to compute it. The integer $ir_X(P)$ is quite useful to get an upper bound for the integer $\alpha(X,P)$.

These two integers $ir_X(P)$ and $\alpha(X,P)$ are the key definitions introduced in this paper. We also add other related numerical invariants related to $ir_X(P)$ and $\alpha(X,P)$. We will see in the proofs that these invariants are quite useful to compute $ir_X(P)$ and $\alpha(X,P)$. First of all, several times it is important to look at zero-dimensional subschemes, not just finite sets, to take the linear span. This was a key ingredient for the study of binary forms ([14], [8], §3, [20], §4) and it is very useful also for multivariate polynomials ([8]). The *cactus rank* $z_X(P)$ of P with respect to X is the minimal degree of a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z\rangle$ ([10], [9]). Let $iz_X(P)$ be the minimal integer t such that there are zero-dimensional subschemes $Z_i \subset X$, $i \ge 1$, such that $\{P\} = \cap_i \langle Z_i\rangle$. Obviously $iz_X(P) \le ir_X(P)$ and $iz_X(P) = 1$ if and only if $P \in X$. Let $\gamma(X,P)$ be the minimal integer x such that there are finitely many zero-dimensional schemes $Z_i \subset X$, say $Z_1,\dots,Z_k$, such that $\{P\} = \cap_{i=1}^k \langle Z_i\rangle$ and $\sum_{i=1}^k \deg(Z_i) = x$. Obviously

$$P \in X, \Leftrightarrow \alpha(X,P) = \Leftrightarrow \gamma(X,P) = 1.$$

Most of our results are for curves and Veronese varieties (in the latter case the X-rank of P is called the symmetric tensor rank of X) (see [2],[8],[15],[19],[20]). In the case of Veronese varieties we give a complete classification of the possible integers $ir_X(P)$, $iz_X(P)$ and $\alpha(X,P)$ when either P has border rank 2 (Theorem 4) or $r_X(P) = 3$ (Theorem 5).

We prove the following results.

Proposition 1. *Let $X \subset \mathbb{P}^{2k}$, $k \ge 1$, be an integral and non-degenerate curve. For a general $P \in \mathbb{P}^{2k}$ we have $r_X(P) = ir_X(P) = k+1$ and $\alpha(X,P) = 2k+2$.*

Theorem 1. *Assume char($\mathbb{K}$) = 0. Let $X \subset \mathbb{P}^{2k+1}$ be an integral and non-degenerate curve. Fix a general $P \in \mathbb{P}^{2k+1}$.*

(a) If X is not a rational normal curve, then $r_X(P) = ir_X(P) = k+1$ and $\alpha(X,P) = 2k+2$.

(b) If X is a rational normal curve, then $r_X(P) = z_X(P) = k+1$, $ir_X(P) = iz_X(P) = k+2$ and $\alpha(X,P) = \gamma(X,P) = 2k+3$.

We also have a result on strange curves (Proposition 3), results on space curves (Theorems 2 and 3) and on rational normal curves (Propositions 5 and 6).

2. Arbitrary characteristic

For any integral variety $X \subset \mathbb{P}^n$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^n$ of the union of all linear spaces $\langle S\rangle$ with $S \subset X$ and $\sharp(S) = t$. Each $\sigma_t(X)$ is an integral variety, $\sigma_1(X) = X$ and $\dim(\sigma_t(X)) \le \min\{n, t\cdot\dim(X) - 1\}$. For each $P \in \mathbb{P}^n$ the X-border rank $b_X(P)$ of X is the minimal integer t such that $P \in \sigma_t(X)$. Let $\tau(X) \subseteq \mathbb{P}^n$ denote the tangent developable of X, i.e. the closure in $\mathbb{P}^n$ of all tangent spaces $T_QX \subseteq \mathbb{P}^n$, $Q \in X_{\mathrm{reg}}$. The algebraic set $\tau(X)$ is an integral variety,

$$\dim(\tau(X)) \le \min\{n, 2\cdot\dim(X)\}$$

and $\tau(X) \subseteq \sigma_2(X)$ (it is called the tangent developable of X).

Notation 1. For any linear subspace $V \subseteq \mathbb{P}^n$ let $\ell_V : \mathbb{P}^n \setminus V \to \mathbb{P}^{n-k-1}$, $k := \dim(V)$, denote the linear projection from V. If V is a single point, O, we often write ℓ_O instead of $\ell_{\{O\}}$.

Notation 2. Let $\mathcal{Z}(X,P)$ (resp. $\mathcal{S}(X,P)$) denote the set of all zero-dimensional schemes $Z \subset X$ (resp. finite sets $S \subset X$) such that $\deg(Z) = z_X(P)$ (resp. $\sharp(S) = r_X(P)$) and $P \in \langle Z\rangle$ (resp. $P \in \langle S\rangle$).

As in [11], Lemma 2.1.5, and [8], Proposition 11, we use the following important invariant $\beta(X)$ of the embedded variety $X \subset \mathbb{P}^n$.

Notation 3. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Let $\beta(X)$ denote the maximal integer t such that any zero-dimensional scheme $Z \subset X$ with $\deg(Z) \le t$ is linearly independent, i.e. $\dim(\langle Z\rangle) = \deg(Z) - 1$.

Remark 1. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. Fix $P \in \mathbb{P}^n$. If $b_X(P) \le \beta(X)$ and X is either a smooth curve or a smooth surface, then $z_X(P) = b_X(P)$ ([11], Lemma 2.1.5, or [8], Proposition 11).

Take any integral and non-degenerate variety $X \subset \mathbb{P}^n$ and any finite set $S \subset X$ such that $\sharp(S) \le \beta(X)$. By the definition of $\beta(X)$ the set S is linearly independent. It seems better in Notation 3 to prescribe the linearly independence of an arbitrary zero-dimensional scheme $Z \subset X$ with $\deg(Z) \le \beta(X)$. Anyway, in many important cases (e.g. the Veronese varieties) the set-theoretic definition and the scheme-theoretic one chosen in Notation 3 give the same integer.

Remark 2. Obviously $\beta(X) \le n+1$ and equality holds if X is a rational normal curve. We claim that equality holds if and only if X is a rational normal curve. Indeed, if X is a curve with degree $d \ge n+1$, then a general hyperplane section of X contains d points spanning only a hyperplane. Now assume $\dim(X) \ge 2$. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Since $H \cap X$ is infinite, we may find $S \subset H \cap X$ with $\sharp(S) = n+1$. Since S is linearly dependent, $\beta(X) \le n$ even in this case.

Remark 3. Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$ and $P \in \mathbb{P}^n$. Obviously $ir_X(P) = +\infty$ if and only if $ir_X(P) > n$. Since the intersection of $n-1$ hyperplanes of $\mathbb{P}^n$ contains at least a line, if $r_X(P) = ir_X(P) = n$, then $\alpha(X,P) = n^2$. We have $r_X(P) = n+1$ if and only if $\dim(X) = 1$ and X is a flat curve in the sense of [4]. Obviously if $r_X(P) = n+1$, then $ir_X(P) = +\infty$. See [4], Proposition 1 and Example 1, for two classes of flat curves.

Let $X \subsetneq \mathbb{P}^n$ be an integral and non-degenerate variety and $P \in \mathbb{P}^n$. We say that P is a *strange point* of X if for a general $Q \in X_{\mathrm{reg}}$ the Zariski tangent space T_QX contains P (we allow the case in which X is a cone with vertex containing P). The *strange set* of X is the set of all strange points of X (this set is always a linear subspace, but usually it is empty). If this set is not empty, then either $\mathrm{char}(\mathbb{K}) > 0$ or X is a cone and the strange set of X is the vertex of X ([7],[22]). Lines and smooth conics in characteristic two are the only smooth strange curves ([17], Theorem IV.3.9). Now fix $P \in \mathbb{P}^n \setminus X$ and set $f_{P,X} := \ell_P|X$. Since $P \notin X$, $f_{P,X}$ is a finite morphism and we have $\deg(X) = \deg(f_{P,X}) \cdot \deg(f_{P,X}(X))$. The point P is a strange point of X if and only if $f_{P,X}$ is not separable. We recall that a non-degenerate curve $X \subset \mathbb{P}^n$, $n \ge 3$, is said to be *very strange* if a general hyperplane section of X is not in linearly general position ([22]). A very strange curve is strange ([22], Lemma 1.1).

Proposition 2. *Fix an integral and non-degenerate variety* $X \subsetneq \mathbb{P}^n$*. Set* $m := \dim(X)$ *and fix* $P \in \mathbb{P}^n$*. If* P *is not a strange point of* X*, then* $ir_X(P) \le n-m+1$*.*

Proof. We will follow the proof of part (a) of [4], Theorem 1. If $P \in X$, then $ir_X(P) = 1$. Hence we may assume $P \notin X$. First assume $m = 1$. Let $H \subset \mathbb{P}^n$ be a general hyperplane containing P. Since P is not a strange point of X, H is transversal to X, i.e. $H \cap \mathrm{Sing}(X) = \emptyset$ and $\sharp(X \cap H) = \deg(X)$. Since X is reduced and irreducible, we have $h^1(\mathcal{I}_X) = 0$. From the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H,H}(1) \to 0 \tag{1}$$

we get that the set $H \cap X$ spans H. Since $P \in H$, we get the existence of $S_H \subset X \cap H$ such that $\sharp(S_H) \le n$ and $P \in \langle S_H \rangle$. Fix general hyperplanes H_i, $i \le i \le n$, containing P and such that $\{P\} = H_1 \cap \cdots \cap H_n$. Take $S_{H_i} \subset X \cap H_i$ as above. Since $\{P\} = \cap_{i=1}^n \langle S_{H_i} \rangle$, we get $ir_X(P) \le n$. Now assume $m \ge 2$. We use induction on m. Take a general hyperplane $H \subset \mathbb{P}^n$ containing P. Bertini's theorem gives that $X \cap H$ is geometrically integral ([18], part 4) of Th. I.6.3). Fix a general $Q \in (X \cap H)_{\mathrm{reg}}$. For general H we may take as Q a general point of X. Hence $P \notin T_Q X$. Hence $P \notin (T_Q X) \cap H = T_Q(X \cap H)$. Thus P is not a strange point of $X \cap H$. By the inductive assumption in $H \cong \mathbb{P}^{n-1}$ we get $ir_{X\cap H}(P) \le n-m+1$. Since $ir_X(P) \le ir_{X\cap H}(P)$, we are done. □

Proposition 3. *Fix an integral and non-degenerate strange curve* $X \subset \mathbb{P}^n$*. Fix* $P \in \mathbb{P}^n \setminus X$ *and assume that* P *is the strange point of* X*. Let* s *(resp.* p^e*) denote the separable (resp. inseparable) degree of* $f_{P,X}$*. Set* $d := \deg(X)$ *and* $c := \deg(f_{P,X}(X))$*. We have* $d = sp^e c$*.*

(a) If $s \ge 2$*, then* $ir_X(P) = 2$*.*

(b) If $s = 1$*,* $c \ne n-1$ *and* X *is not very strange, then* $ir_X(P) \le n$*.*

(c) If $s = 1$ *and* $c = n-1$*, then* $r_X(P) = n+1$ *and* $ir_X(P) = +\infty$*.*

Proof. Since $P \notin X$, $f_{P,X}$ is a finite morphism. Hence $\deg(X) = \deg(f_{P,X}) \cdot \deg(f_{P,X}(X))$, i.e. $d = sp^e c$.

First assume $s \ge 2$. Fix general $P_1, P_2 \in f_{P,X}(X)$. By assumptions there are $O_{ij} \in f_{P,X}^{-1}(P_i)$, $i = 1,2$, $j = 1,2$, such that $O_{i1} \ne O_{i2}$. Set $S_i := \{O_{i1}, O_{i2}\}$. Since $P \in \langle S_i \rangle$, $i = 1,2$, and the two lines $\langle S_i \rangle$ are different, we get $ir_X(P) = 2$.

From now on we assume $s = 1$ and that X is not very strange. Let $u : Y \to X$ denote the normalization map. Let $\mathcal{H}$ be the set of all hyperplanes of $\mathbb{P}^{n-1}$ transversal to $f_{P,X}(X)$. We have $\dim(\mathcal{H}) = n-1$. Since $f_{P,X}(X)$ is non-degenerate, we have $\deg(f_{P,X}(X)) \ge n-1$.

First assume $c \ne n-1$. Hence for every $H \in \mathcal{H}$ we may find a set $A_H \subset H \cap f_{P,X}(X)$ such that $\sharp(A_H) = n$ and $\langle A_H \rangle = H$. Notice that A_H is linearly dependent. Fix $S_H \subset X$ such that $\sharp(S_H) = n$ and $f_{P,X}(S_H) = A_H$. If $P \notin \langle S_H \rangle$, then S_H is linearly dependent. Since X is not very strange, we have $X \cap \langle S \rangle = S$ (as sets) for a general set $S \subset X$ such that $\sharp(S) = n-1$. Hence there is at most an $(n-2)$-dimensional family of linearly dependent subsets of X with cardinality n. Hence there is a non-empty open subset $\mathcal{H}'$ of $\mathcal{H}$ such that $P \in \langle S_H \rangle$ for every $H \in \mathcal{H}'$. Since $\cap_{H\in\mathcal{H}'} H = \emptyset$, we get $\{P\} = \cap_{H\in\mathcal{H}'} \langle S_H \rangle$. Hence $ir_X(P) \le n$.

Now assume $c = n-1$. Hence $f_{P,X}(X)$ is a rational normal curve. In particular $f_{P,X}(X)$ is smooth. Since $f_{P,X} \circ u : Y \to f_{P,X}(X)$ is a purely inseparable morphism between smooth curves, it is injective. Hence $f_{P,X}$ is injective. Since $f_{P,X}(X)$ is a rational normal curve, for every $S \subset X$ with $\sharp(S) \le n$, the set $f_{P,X}(S)$ is a linearly independent set with $\sharp(S)$ elements. Hence $P \notin \langle S\rangle$. Hence $r_X(P) = n+1$. Hence $ir_X(P) > n$, i.e. $ir_X(P) = +\infty$. □

All strange curves may be explicitly constructed (see [7] for the case $n = 2$ and [3] for the case $n > 2$).

3. Curves

We use the following obvious observations (true in arbitrary characteristic) and whose linear algebra proof is left to the reader (parts (a) and (b) of Lemma 1 just say that two distinct lines have at most one common point and that if $P \in \langle\{P_1, P_2\}\rangle$ and $ir_X(P) < 4$, then there is $S \subset X$ with $\sharp(S) \le 3$, $P \in \langle S\rangle$ and $\langle\{P_1, P_2\}\rangle \nsubseteq \langle S\rangle$).

Lemma 1. *Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Fix $P \in \mathbb{P}^3 \setminus X$.*
(a) If $r_X(P) = ir_X(P) = 2$, then $\alpha(X, P) = 4$.
(b) If $r_X(P) = 2$ and $ir_X(P) = 3$, then $\alpha(X, P) = 5$.
(c) If $r_X(P) = ir_X(P) = 3$, then $\alpha(X, P) = 9$.

Remark 4. Now assume that X is a singular curve, but take a zero-dimensional scheme $Z \subset X_{\mathrm{reg}}$ such that $k := \deg(Z) \le \beta(X)/2$. Since Z is curvilinear, it has finitely many linear subschemes. Since Z is linearly independent, the set $\Psi := \langle Z\rangle \setminus_{Z' \subsetneq Z} \langle Z'\rangle)$ is a non-empty open subset of the $(k-1)$-dimensional linear space $\langle Z\rangle$. Fix any $P \in \Psi$. Lemma 3 gives $z_X(P) = k$ and that Z is the only degree k subscheme of X whose linear span contains P. Since $Z \subset X_{\mathrm{reg}}$, Z is smoothable. Hence [8], Proposition 11, give $b_X(P) = k$.

Lemma 2. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Fix $P \in \mathbb{P}^n$ such that $z_X(P) \le \beta(X)/2$. Then:*
(i) *There is a unique zero-dimensional scheme $A \subset X$ such that $P \in \langle A\rangle$ and $\deg(A) \le z_X(P)$. We have $\deg(A) = z_X(P)$.*
(ii) *Fix any zero-dimensional scheme $W \subset X$ such that $\deg(W) \le \beta(X) - z_X(P)$ and $P \in \langle W\rangle$. Then $W \supseteq A$. We have $ir_X(P) \ge iz_X(P) \ge \beta(X) - z_X(P) + 1$.*
(iii) *Assume that A is not reduced. Then $r_X(P) \ge \beta(X) - z_X(P) + 1$. If $r_X(P) = \beta(X) - z_X(P) + 1$, then $S \cap A = \emptyset$ for all sets $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S\rangle$.*

Proof. Assume the existence of zero-dimensional schemes A, W such that $A \ne W$, $P \in \langle A\rangle \cap \langle W\rangle$, $P \notin \langle A'\rangle$ for all $A' \subsetneq A$ and $\deg(A) + \deg(W) \le \beta(X)$. Lemma 3 gives the existence of $W' \subsetneq W$ such that $P \in \langle W'\rangle$. If $W' \ne W$, then we continue taking W' instead of W. We get parts (a) and (b).

The first assertion of part (iii) follows from part (ii), while the second one follows from Lemma 3. □

Proposition 4. *Let $X \subset \mathbb{P}^3$ be a rational normal curve. Then $ir_X(P) = 3$ for all $P \in \mathbb{P}^3 \setminus X$.*

Proof. Lines and smooth conics in characteristic two are the only smooth strange curves ([17], Theorem IV.3.9). Fix $P \in \mathbb{P}^3 \setminus X$. Since X is not strange, we have $ir_X(P) \le 3$ (Proposition 3) (even in positive characteristic). Since $\sigma_2(X) = \mathbb{P}^3$ ([1], Remark 1.6), Remark 3 gives $z_X(P) = 2$. Since $\beta(X) = 4$, Lemma 3 gives $ir_X(P) \ge 3$. □

Let X be a smooth elliptic curve defined over $\mathbb{K}$. We recall that the 2-rank of X is the number, ϵ, of pairwise non-isomorphic line bundles L on X such that $L^{\otimes 2} \cong \mathcal{O}_X$ ([23], Chapter III). If $\mathrm{char}(\mathbb{K}) \neq 2$, then $\epsilon = 4$, while $\epsilon \in \{1, 2\}$ if $\mathrm{char}(\mathbb{K}) = 2$ ([23], Corollary III.6.4).

Theorem 2. *Let $X \subset \mathbb{P}^3$ be a smooth elliptic curve. Fix $P \in \mathbb{P}^3 \setminus X$. Let ϵ be the 2-rank of the elliptic curve X. There are exactly ϵ quadric cones W_i, $1 \le i \le \epsilon$ containing X. Call O_i, $1 \le i \le \epsilon$, the vertex of W_i.*

(a) The points O_i, $1 \le i \le \epsilon$, are the only points $Q \in \mathbb{P}^3$ such that $\mathcal{Z}(X, P)$ and $\mathcal{S}(X, Q)$ are infinite; we have $ir_X(O_i) = 2$ for all i; each point O_i is contained in TX.

(b) If $P \in (TX \cup \bigcup_{i=1}^{\epsilon} W_i)$, but $P \neq O_i$ for any i, then $ir_X(P) = 3$.

(c) If $P \notin (TX \cup \bigcup_{i=1}^{\epsilon} W_i)$, then $ir_X(P) = 2$.

Proof. Call R_i, $1 \le i \le \epsilon$, the pairwise non-isomorphic line bundles on X such that $R_i^{\otimes 2} \cong \mathcal{O}_X$. Since $\deg(X)$ is even and $\mathbb{K}$ is algebraically closed, there is a line bundle $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X(1)$. Set $L_i := R_i \otimes \mathcal{L}$. It is easy to check that the line bundles L_i, $1 \le i \le \epsilon$, are pairwise non-isomorphic and that, up to isomorphisms, they are the only line bundles A on X such that $A^{\otimes 2} \cong \mathcal{O}_X(1)$.

Since X is not strange, Proposition 3 gives $ir_X(P) \le 3$. Since $P \notin X$, Remark 3 and [1], Remark 1.6, give $z_X(P) = 2$. Obviously, if $\sharp(\mathcal{Z}(X, P)) = 1$, then $ir_X(P) > 2$. Since $\ell_P(X)$ spans $\mathbb{P}^2$, we have $\deg(\ell_P(X)) \ge 2$. Hence either $\deg(\ell_P(X)) = 4$ and $\ell_P|X$ is birational onto its image or $\deg(\ell_P|X) = 2$.

First assume $\deg(\ell_P|X) = 2$. In this case we get that $\mathcal{Z}(X, P)$ is infinite. Since $\ell_P(X) \cong \mathbb{P}^1$, the morphism $\ell_P|X$ is not purely inseparable. Hence a general fiber of it is formed by two distinct points of X spanning a line through P. Hence $ir_X(P) = 3$. We get $\mathcal{O}_X(1) \cong \ell_P(\mathcal{O}_{\ell_P(X)}(1))$. Since $\mathcal{O}_{\ell_P(X)}(1) \cong R^{\otimes 2}$ with R a degree 1 line bundle on $\ell_P(X)$, $\ell_P^*(R)$ is one of the line bundle L_i, $1 \le i \le \epsilon$. Since $X \neq \mathbb{P}^1$, $\ell_P|X$ has at least one ramification point. Hence $O_i \in TX$ for all i. The construction may be inverted in the following sense. Fix one of the line bundles L_i, $1 \le i \le \epsilon$. Since X is an elliptic curve, we have $h^0(X, L_i) = 2$ and the linear map $j : S^2(H^0(X, L_i)) \to H^0(X, \mathcal{O}_X(1))$ is injective with as image a hyperplane of the 4-dimensional linear space $H^0(X, \mathcal{O}_X(1))$, i.e. (by the linear normality of X) a point, $\widetilde{O}_i$ of $\mathbb{P}^3 = \mathbb{P}(H^0(X, \mathcal{O}_X(1))^\vee)$. The definition of j gives that $\ell_{\widetilde{O}_i}|X$ has degree 2.

Now assume $\deg(\ell_P(X)) = 4$. The genus formula for plane curves gives that $\ell_P(X)$ has 1 or 2 singular points and that if it has two singular points, then they are either ordinary nodes or ordinary cusps. If $\ell_P(X)$ has either a unique singular point or at least one cusp, then $ir_X(P) > 2$ and hence $ir_X(P) = 3$. In particular this is the case if $P \in TX$. Hence if $P \in TX$ and $P \neq O_i$, then $ir_X(P) = 3$. Now assume $P \notin TX$. In this case $ir_X(P) = 2$ if and only if $\ell_P(X)$ has two singular points. If the plane curve $\ell_P(X)$ has a unique singular point, then it is an ordinary tacnode. Let $T \subset \mathbb{P}^3$ be a line secant to X, but not tangent to X. Since X is the complete intersection of two quadric surfaces, there is a unique quadric surface, W, containing $X \cup \{P\}$. Call T a line in W containing P. $X \cup T$ is contained in a unique quadric surface, W. If W is singular, i.e. if $W = W_i$ for some i, then there is a unique line through P and secant to X. If W is smooth, i.e. if $P \notin W_i$ for any i, then there are two such lines, both of them containing two distinct points of X, because we assumed $P \notin TX$. Hence $ir_X(P) = 2$ in this case. □

Theorem 3. *Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Assume that X is not strange and that X has only planar singularities. There is a non-empty open subset Ω of $\mathbb{P}^3 \setminus X$ such that $ir_X(P) = 2$ for all $P \in \Omega$ if and only if X is not a rational normal curve..*

Proof. Set $d := \deg(X)$ and $q := p_a(X)$. Since Proposition 4 gives that " only if " part, it is sufficient to prove the " if " part. Assume $d \geq 4$. It is easy to check the existence of a non-empty open subset W of $\mathbb{P}^3 \setminus X$ such that $\ell_P|X$ is birational onto its image for all $P \in W$. By assumption for each $O \in \mathrm{Sing}(X)$ the Zariski tangent plane T_OX of X at O is a plane. Since $\mathrm{Sing}(X)$ is finite, we get finitely many planes T_OX, $O \in \mathrm{Sing}(X)$, and we call W' the intersection of W with the complement of the union of these planes. Let G be the intersection of W' with the complement of the tangent developable $\tau(X)$ of X. For each $P \in G$ the morphism $\ell_P|X$ is unramified and birational onto its image. Hence the singularities of the degree d plane curve $\ell_P(X)$ comes only from the non-injectivity of $\ell_P|X$ and the singularities of X. To prove Theorem 3 it is sufficient to prove that the set of all $P \in G$ such that $\ell_P|X$ has at least two fibers with cardinality ≥ 2 contains a non-empty open subset. For any $O \in \mathrm{Sing}(X)$ let $C_O(X)$ the cone with vertex O and the plane curve $\overline{\ell_O(X \setminus \{O\})}$ as its base. Set $G' := G \setminus G \cap (\cup_{O \in \mathrm{Sing}(X)} C_O(X))$. The set G' is a non-empty open subset of G and for every $P \in G'$ no point of $X \setminus \mathrm{Sing}(X)$ is mapped onto a point of $\ell_P(\mathrm{Sing}(X))$. Hence for each $P \in G'$ the plane curve $\ell_P(X)$ has $\sharp(\mathrm{Sing}(X))$ singular points isomorphic to the corresponding singular points of X, plus some other singular points and the integer $p_a(\ell_P(X)) - q = (d-1)(d-2)/2 - q$ is the sum of the contributions of the other singular points. Since X is not strange, it is not very strange, i.e. a general secant line of X contains only two points of X ([22], Lemma 1.1). This is equivalent to the existence of a non-empty open subset G'' of G' such that for all $P \in G''$ each singular point of $\ell_P(X) \setminus \ell_P(\mathrm{Sing}(X))$ has only two branches.

Claim: There is a non-empty open subset G_1 of G'' such that for every $P \in G_1$, $\ell_P(X) \setminus \ell_P(\mathrm{Sing}(X))$ has only ordinary double points as singularities.

Proof of the Claim: Fix $P \in G''$. Fix $O \in \ell_P(X) \setminus \ell_P(\mathrm{Sing}(X))$. By the definition of G'' there are exactly two points $Q_1, Q_2 \in X$ such that $\ell_P(Q_1) = \ell_P(Q_2) = O$, X is smooth at Q_1 and Q_2, and $\ell_P|X$ is unramified at each Q_i. Hence $\ell_P(X) \setminus \ell_P(\mathrm{Sing}(X))$ has only ordinary double points as singularities if and only if $\ell_P(T_{Q_1}X) \neq \ell_P(T_{Q_2}X)$, i.e. if and only if the planes $\langle \{P\} \cup T_{Q_i}X \rangle$, $i = 1, 2$, are distinct. This is certainly true if $T_{Q_1}X \cap T_{Q_2}X = \emptyset$. Let $\mathcal{V}$ denote the set of all $(Q_1, Q_2) \in (X \setminus \mathrm{Sing}(X)) \times (X \setminus \mathrm{Sing}(X))$ such that $Q_1 \neq Q_2$. Let $\mathcal{U}$ be the set of all $(Q_1, Q_2) \in \mathcal{V}$ such that $T_{Q_1}X \cap T_{Q_2}X \neq \emptyset$. Since X is not strange, $\mathcal{U}$ is a union of finitely many subvarieties of dimension ≤ 1; it is here that we use the full force of our assumption " X not strange ", not only the far weaker condition " X not very strange ". Let Δ be the closure in $\mathbb{P}^3$ of the union of the lines $\langle \{Q_1, Q_2\} \rangle$ with $(Q_1, Q_2) \in \mathcal{U}$. We have $\dim(\Delta) \leq 2$. Set $G_1 := G'' \cap (\mathbb{P}^3 \setminus \Delta)$. By construction this set G_1 satisfies the Claim.

Now we prove that we may take $\Omega := G_1$. Fix $P \in G_1$ and call x the number of the singular points of $\ell_P(X) \setminus \ell_P(\mathrm{Sing}(X))$. By the claim it is sufficient to prove the inequality $x \geq 2$. Since $\ell_P(X)$ is a plane curve of degree d, it has arithmetic genus $(d-1)(d-2)/2$. Since each point of $\ell_P(X) \setminus \ell_P(\mathrm{Sing}(X))$ is an ordinary node, $\ell_P|X$ is unramified at each point of $\mathrm{Sing}(X)$ and $\ell_P^{-1}(\ell_P(X) \setminus \ell_P(\mathrm{Sing}(X)))$, we have $x = p_a(\ell_P(X)) - p_a(X) = (d-1)(d-2)/2 - q$. Hence it is sufficient to prove that $q \le (d-1)(d-2)/2 - 2$. This is true by the assumption $d \ge 4$ and Castelnuovo's inequality for the arithmetic genus of space curves (use [22], Lemma 1.1, that X is not strange and that the upper bound needs only that a general plane section of X is in linearly general position). $\square$

Proof of Proposition 1: Let Δ denote the set of all linearly independent subsets of X with cardinality $k+1$. Since $\sigma_{k+1}(X) = \mathbb{P}^{2k}$ and $\dim(\sigma_k(X)) = 2k-1$ ([1], Remark 1.6), we have $r_X(P) = k+1$. A dimensional count gives that $\mathcal{S}(X,P)$ has a one-dimensional irreducible component, Γ. Fix $A, B \in \Gamma$. It is sufficient to prove that $\{P\} = \langle A\rangle \cap \langle B\rangle$. Since any two k-dimensional linear subspaces meet, the set A may be seen as a general element of Δ and, after fixing A, P may be seen as a general element of $\langle A\rangle$. Hence it is sufficient to prove that $\langle A\rangle \cap \langle B\rangle$ is a single point for a general $(A,B) \in \Delta \times \Delta$, i.e. to check that $A \cup B$ spans $\mathbb{P}^{2k}$. For fixed A, we have $\langle A \cup B\rangle = \mathbb{P}^{2k}$ for a general $B \subset X$, because X spans $\mathbb{P}^{2k}$. $\square$

Proof of Theorem 1: Since $\sigma_{k+1}(X) = \mathbb{P}^{2k+1}$ and P is general, we have $r_X(P) \le k+1$ ([1], Remark 1.6). Since $\dim(\sigma_k(X)) = 2k-1$ ([1], Remark 1.6) and P is general, we have $r_X(P) \ge k+1$. Hence $r_X(P) = k+1$. X is not a rational normal curve if and only if there are $S_1, S_2 \subset X$ such that $S_1 \ne S_2$, $\sharp(S_1) = \sharp(S_2) = k+1$ and $P \in \langle S_1\rangle \cap \langle S_2\rangle$ ([13], Theorem 3.1). Let Ω be the set of all $Q \in \mathbb{P}^{2k+1} \setminus \sigma_k(X)$ such that there are only finitely many sets $S \subset X$ with $\sharp(S) = k+1$ and $Q \in \langle S\rangle$. Ω is a non-empty open subset of $\mathbb{P}^{2k+1}$. Since P is general, we may assume $P \in \Omega$.

(i) In this step we assume that X is not a rational normal curve. Let Γ denote the set of all finite sets $S \subset X$ such that $\sharp(S) = k+1$ and $\dim(\langle S\rangle) = k$. We proved the existence of $S_i \in \Gamma$, $i = 1,2$, such that $P \in \langle S_1\rangle \cap \langle S_2\rangle$. To prove part (a) it is sufficient to prove that $\{P\} = \langle S_1\rangle \cap \langle S_2\rangle$ for a general P. Assume that this is not true, i.e. assume that $\langle S_1\rangle \cap \langle S_2\rangle$ is a linear space of dimension $\rho > 0$. Notice that $\mathcal{S}(X,P) = \{S \in \Gamma : P \in \langle S\rangle\}$. Set $\Gamma(S_1) := \{S \in \Gamma : S \cap S_1 = \emptyset, \langle S\rangle \cap \langle S_1\rangle \cap \Omega \ne \emptyset\}$. Since $\dim\langle S_1\rangle = k$ and $P \in \Omega \cap \langle S_1\rangle$, then $\Gamma(S_1) \ne \emptyset$ and $\Gamma(S_1)$ has pure dimension k. Since P is general in $\mathbb{P}^{2k+1}$, we may assume that S_1 is general in Γ and that S_2 is general in one of the irreducible components of $\Gamma(S_1)$. We get that for a general $P' \in \Omega \cap \langle S_1\rangle$ there is a ρ-dimensional family of sets S with $P' \in \langle S\rangle$, absurd.

(ii) In this step we assume that X is a rational normal curve. We know that $r_X(P) = k+1$. We proved that $ir_X(P) \ge k+2$ and hence that $\alpha(X,P) \ge 2k+3$. For a sufficiently general $P \in \mathbb{P}^{2k+1}$ we call S_P the only subset of X with cardinality $k+1$ and whose linear span contains P. Since $\beta(X) = 2k+2$ and $P \notin \sigma_k(X)$, Remark 3 gives $z_X(P) = k+1$ and that S_P is the only degree $k+1$ zero-dimensional subscheme of X whose linear span contains P. Hence $iz_X(P) \ge k+2$ and $\gamma(X,P) \ge 2k+3$.

Fix a general $Q \in X$ and let $\phi : X \to \mathbb{P}^{2k}$ denote the morphism induced from $\ell_Q|(X \setminus \{Q\})$. The morphism ϕ is an embedding of $X \cong \mathbb{P}^1$ as a rational normal curve of $\mathbb{P}^{2k}$. Fix a general $P' \in \mathbb{P}^{2k}$. Proposition 1 gives the existence of $A_1, A_2 \subset \phi(X)$ such that $\sharp(A_1) = \sharp(A_2) = k+1$ and $\langle A_1\rangle \cap \langle A_2\rangle = \{P'\}$. For a fixed point $\phi(Q)$, but for general P' we may also assume $\phi(Q) \notin (A_1 \cup A_2)$. Hence there is a unique set $B_i \subset X \setminus \{Q\}$ such that $\phi(B_i) = A_i$. Set $E_i := \{Q\} \cup B_i$. Fix $P'' \in \mathbb{P}^{2k+1}$ such that $\ell_Q(P'') = P'$. For fixed Q, but general P' we may consider

P'' as a general point of $\mathbb{P}^{2k+1}$. We have $\langle\{Q, P''\}\rangle = \langle E_1\rangle \cap \langle E_2\rangle$. Varying Q in X we get $ir_X(P) \le k+2$ and hence $ir_X(P) = k+2$. Let Θ be the set of all finite subsets $A \subset X$ such that $\sharp(A) = k+2$ and $P \in \langle A\rangle$. Assume for the moment the existence of $A \in \Theta$ such that $A \cap S_P = \emptyset$, i.e. such that $\sharp(A \cup S_P) = 2k+3$. Since $\beta(X) = 2k+2$ and $\sharp(A \cup S_P) = 2k + 3$, we get $\langle S_P \cup A\rangle = \mathbb{P}^{2k+1}$, i.e. $\dim(\langle A\rangle \cap \langle S_P\rangle) = 0$ (Grassmann's formula). Since $P \in \langle A\rangle \cap \langle S_P\rangle$, we get $\{P\} = \langle A\rangle \cap \langle S_P\rangle$, i.e. $\alpha(X, P) \le 2k + 3$. Hence $\alpha(X, P) = \gamma(X, P) = 2k + 3$. Now assume $A \cap S_P \neq \emptyset$ for all $A \in \Theta$. Since P is general and $\sigma_{k+2}(X) = \mathbb{P}^{2k+1}$, Terracini's lemma (or a dimensional count) gives $\dim(\Theta) = 2$. For any $Q \in S_P$ set $\Theta_Q := \{A \in \Theta : Q \in A\}$. The proof of the inequality $ir_X(P) \le 2k + 3$ also shows $\dim(\Theta_Q) = 1$. Since S_P is finite, we get $\dim(\Theta) = 1$, a contradiction. □

4. Veronese varieties

For all integers $m \ge 1$ and $d \ge 1$ let $\nu_d : \mathbb{P}^m \to \mathbb{P}^n$, $n := \binom{m+d}{m} - 1$ denote the order d embedding of $\mathbb{P}^m$ induced by the vector space of all degree d homogeneous polynomials in $d + 1$ variables. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$.

We often use the following elementary lemma ([5], Lemma 1).

Lemma 3. *Fix any $P \in \mathbb{P}^n$ and two zero-dimensional subschemes A, B of $\mathbb{P}^n$ such that $A \neq B$, $P \in \langle A\rangle$, $P \in \langle B\rangle$, $P \notin \langle A'\rangle$ for any $A' \subsetneq A$ and $P \notin \langle B'\rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$.*

We first need the case $m = 1$ of Theorem 4, i.e. we need to study the case in which X is a rational normal curve (Propositions 5,6 and 7).

Proposition 5. *Let $X \subset \mathbb{P}^d$, $d \ge 3$, be a rational normal curve. Fix a set $A \subset X$ with $\sharp(A) = 2$ and any $P \in \langle A\rangle \setminus A$. Then $r_X(P) = z_X(P) = 2$, $ir_X(P) = iz_X(P) = d$ and $\alpha(X, P) = \gamma(X, P) = d + 2$. Moreover, there is a set $B \subset X$ such that $\sharp(B) = d$ and $\{P\} = \langle A\rangle \cap \langle B\rangle$.*

Proof. Since $\beta(X) = d + 1 \ge 3$, we have $A = \langle A\rangle \cap X$. Hence $P \notin X$. Hence $ir_X(P) = 2 = iz_X(P)$. Fix a zero-dimensional scheme $W \subset X$ such that $P \in \langle W\rangle$, $P \notin \langle W'\rangle$ for any $W' \subsetneq W$ and $W \neq A$. Since $\beta(X) = d + 1$, Lemma 3 gives $\deg(W) \ge d$. Hence $ir_X(P) \ge iz_X(P) \ge d$ and $\alpha(X, P) \ge \gamma(X, P) \ge d + 2$. Hence to conclude the proof it is sufficient to find a set $B \subset X$ such that $\sharp(B) = d$ and $\{P\} = \langle A\rangle \cap \langle B\rangle$. Set $Y := \ell_P(X)$. Since $P \in \langle A\rangle$ and $P \notin X$, the curve Y is a linearly normal curve with degree d, arithmetic genus 1 and a unique singular point, which is an ordinary node. Fix a general hyperplane $H \subset \mathbb{P}^{d-1}$ and set $E := Y \cap X$. Since H is general, it does not contain the singular point of Y and it is transversal to Y. Hence E is a set of d points and there is $B \subset X$ such that $\sharp(B) = d$ and $\ell_P(B) = E$. Since $\sharp(B) \le \beta(X)$, B is linearly independent. Since E is linearly dependent, we have $P \in \langle B\rangle$. Since $\sharp(A \cup B) = d + 2 = \beta(X) + 1$, we have $\langle A \cup B\rangle = \mathbb{P}^d$. Hence Grassmann's formula gives $\{P\} = \langle A\rangle \cap \langle B\rangle$. □

Proposition 6. *Let $X \subset \mathbb{P}^d$, $d \ge 3$, be a rational normal curve. Fix $P \in \tau(X)\setminus X$, i.e. fix $P \in \sigma_2(X)$ such that $r_X(P) > 2$. Then $z_X(P) = 2$, $iz_X(P) = d$, $\gamma(X, P) = d + 2$, $r_X(P) = d$, $ir_X(P) = d$ and $\alpha(X, P) = d^2$. Moreover, there are a zero-dimensional $A \subset X$ and a finite set $B \subset X$ such that $\deg(A) = 2$, $\sharp(B) = d$ and $\{P\} = \langle A\rangle \cap \langle B\rangle$.*

Proof. First of all we explain the " i.e. " part. Since $\beta(X) \ge 2$, Remark 3 gives that for each $Q \in \sigma_2(X) \setminus X$ there is a degree 2 zero-dimensional scheme $A_Q \subset X$

such that $Q \in \langle A_Q\rangle$. Since $\beta(X) \ge 4$, we also get the uniqueness of A_Q. Hence $P \in \tau(X) \Leftrightarrow A_P$ is not reduced $\Leftrightarrow r_X(P) > 2$. Set $A := A_P$. Lemma 3 gives $r_X(P) \ge d$ and $iz_X(P) \ge d$. We repeat the proof of Proposition 5 (now Y is a degree d linearly normal curve with a cusp). We get the existence of a set $B \subset X$ such that $\sharp(B) = d$ and $\{P\} = \langle A\rangle \cap \langle B\rangle$. Hence $iz_X(P) = d$, $\gamma(X,P) = d$. Since $d \ge 3$, X is not strange. Hence $ir_X(P) \le d$ (Proposition 3). Since $r_X(P) \ge d$, we get $r_X(P) = ir_X(P) = d$. Since $r_X(P) = d$, P is contained in no linear space of dimension $\le d-2$ spanned by a finite subset of X. Hence $\alpha(X,P) = d^2$ (Remark 3). □

Proposition 7. *Let $X \subset \mathbb{P}^d$, $d \ge 5$, be a rational normal curve. Fix a set $A \subset X$ such that $\sharp(A) = 3$ and any $P \in \langle A\rangle$ such that $P \notin \langle A'\rangle$ for any $A' \subsetneq A$. Then $r_X(P) = z_X(P) = 3$, $ir_X(P) = iz_X(P) = d-1$ and $\alpha(X,P) = \gamma(X,P) = d+2$.*

Proof. Since $\beta(X) \ge 5$, Lemma 3 gives $z_X(P) = 3$, $iz_X(P) \ge \beta(X)+1-\sharp(A) = d-1$ and hence $r_X(P) = 3$, $ir_X(P) \ge d-1$, $\alpha(X,P) \ge \gamma(X,P) \ge d+2$.

Set $Y := \ell_P(X)$. Since $\beta(X) = d+1 \ge 5$ and $P \notin \langle A'\rangle$ for any $A' \subsetneq A$, $\ell_P|X$ is an embedding. Hence Y is a smooth rational curve of degree d spanning $\mathbb{P}^{d-1}$. Fix any $E \subset X \setminus A$ with $\sharp(E) = d-4$ and set $F := \ell_P(E)$. Since $\sharp(A \cup E) \le \beta(X)$, F is a set of $d-4$ points of Y spanning a $(d-5)$-dimensional linear subspace disjoint from the line $\langle \ell_P(A)\rangle$.

Claim: For general E we have $\langle F\rangle \cap Y = F$ (as schemes) and $\ell_{\langle F\rangle}|(Y \setminus F)$ extends to an embedding $\phi : Y \to \mathbb{P}^3$ with $\phi(Y) \subset \mathbb{P}^3$ a smooth and rational curve of degree 4 with $\phi(\ell_P(A))$ the union of 3 distinct and collinear points.

Proof of the Claim: The map ϕ is induced by the linear projection of X from the linear subspace $\langle \{P\} \cup E\rangle$. Since $E \cap A = \emptyset$ and $\sharp(E \cup A) \le \beta(X)$, we have $\langle E\rangle \cap \langle A\rangle = \emptyset$. Hence $\phi(A)$ is the union of 3 distinct collinear points. For degree reasons we get $\langle F\rangle \cap Y = F$ (as schemes), i.e. $\deg(\phi)\cdot\deg(\phi(Y)) = \deg(Y) - d + 4 = 4$. Since $\phi(Y)$ spans $\mathbb{P}^3$, we get $\deg(\phi) = 1$. Since $\phi(Y)$ has a 3-secant line, the curve Y is not the complete intersection of two quadric surfaces. Hence $\phi(Y)$ is smooth and rational.

Since $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10 = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)) + 1$, the Claim implies the existence of a quadric surface T containing $\phi(Y)$. Since $\phi(Y)$ has genus $\ne 1$, T is not a cone ([17], V.Ex.2.9). Hence $\phi(Y)$ is a curve of type $(1,3)$ on the smooth quadric surface T. The set $\phi(\ell_P(A))$ is contained in a line of type $(1,0)$. Let G be the intersection of $\phi(Y)$ with a general line of type $(1,0)$ of T. Since any two different lines of T are disjoint, we have $\phi(A) \cap G = \emptyset$. Since $\phi(\ell_P(A))$ is reduced, in arbitrary characteristic we get that G is reduced. Since the set $\phi(F)$ is finite, for a general line of type $(1,0)$ on T we have $G \cap \phi(F) = \emptyset$. Hence there is $G' \subset Y \setminus F$ such that $\phi(G') = G$. Let $B \subset X$ be the only set such that $\ell_P(B) = F \cup G'$. Since $\sharp(B) \le \beta(X)$, we have $\dim(\langle B\rangle) = d-2$. Since G is linearly dependent, $F \cup G'$ is linearly dependent. Hence $P \in \langle B\rangle$. Since $A \cap B = \emptyset$ and $\beta(X) = d+1 \le \sharp(A \cup B)$, we have $\langle A \cup B\rangle = \mathbb{P}^d$. Hence Grassmann's formula gives that $\langle A\rangle \cap \langle B\rangle$ is a single point. Hence $\{P\} = \langle A\rangle \cap \langle B\rangle$. Hence $ir_X(P) \le d-1$ and $\alpha(X,P) \le d+2$. Since we proved the opposite inequalities, we are done. □

Theorem 4. *Fix integers $m \ge 1$ and $d \ge 3$. Set $n := n_{m,d} := \binom{m+d}{m} - 1$ and $X := X_{m,d}$. Fix $P \in \sigma_2(X_{m,d}) \setminus X$.*

(a) Assume $P \notin \tau(X)$, i.e. assume $r_X(P) = 2$. Then $ir_X(P) = d$, $z_X(P) = 2$, $iz_X(P) = d$ and $\alpha(X,P) = \gamma(X,P) = d+2$

(b) Assume $P \in \tau(X)\backslash X$. Then $z_X(P) = 2$, $iz_X(P) = ir_X(P) = d$, $\gamma(X,P) = d+2$. If $m = 1$, then $\alpha(X,P) = d^2$. If $m \ge 2$, then $\alpha(X,P) = 3d$.

Proof. Since $d \ge 3$, we have $\sigma_2(X) \ne \tau(X)$, $\sigma_2(X) \setminus \tau(X) = \{P \in \sigma_2(X) : r_X(P) = 2\}$ and $r_X(P) = d$ for each $P \in \tau(X) \setminus X$ ([8], Theorem 32). Since the case $m = 1$ is true (Propositions 5 and 6), we assume $m \ge 2$. Since $\beta(X) = d+1$ (e.g. by [8], Lemma 34), Remark 3 and Lemma 3 imply the existence of a unique zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = 2$ and $P \in \langle Z \rangle$. We have $r_X(P) = 2$ if and only if Z is reduced. Let $A \subset \mathbb{P}^m$ be the degree 2 zero-dimensional scheme such that $\nu_d(A) = Z$. Let $L \subset \mathbb{P}^m$ be the line spanned by A. Set $R := \nu_d(L)$. Since $Z \subset R$, we have $r_X(P) \le r_R(P)$, $z_X(P) \le z_R(P)$, $ir_X(P) \le ir_R(P)$, $iz_X(P) \le iz_R(P)$, $\alpha(X,P) \le \alpha(R,P) = d$ and $\gamma(X,P) \le \gamma(R,P)$. Propositions 5 and 6 give $ir_R(P) = iz_R(P) = d$ and $\gamma(R,P) = d+2$. Let $W \subset \mathbb{P}^m$ be a zero-dimensional scheme such that $P \in \langle \nu_d(W) \rangle$, $P \notin \langle \nu_d(W') \rangle$ for any $W' \subsetneq W$ and $W \ne A$. Since $\beta(X) \ge d+1$, Lemma 3 gives $\deg(W) \ge d$. Hence $iz_X(P) \ge d$ and $\gamma(X,P) \ge d+2$. Hence $ir_X(P) = iz_X(P) = d+2$ and $\gamma(X,P) = d+2$. In case (a) we have $\alpha(X,P) = d+2$, because $\alpha(R,P) = d+2$ (Proposition 5). Now assume that Z is not reduced, i.e. assume $P \in \tau(X)$. Let $C \subset \mathbb{P}^m$ be a smooth conic containing A. The curve $\nu_d(C)$ is a degree $2d$ rational normal curve in its linear span. Since $P \in \langle Z \rangle \subset \langle \nu_d(C) \rangle$, the " Moreover " part of Proposition 6 applied to $\nu_d(C)$ gives the existence of a set $B \subset C$ such that $\sharp(B) = 2d$ and $\langle Z \rangle \cap \langle \nu_d(B) \rangle = \{P\}$. Let $M \subseteq \mathbb{P}^m$ be the plane containing $C \cup L$. Since the restriction maps $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d)) \to H^0(M, \mathcal{O}_M(d))$ and $H^0(M, \mathcal{O}_M(d)) \to H^0(T, \mathcal{O}_T(d))$ are surjective for $T = L$, $T = C$, and $T = C \cup L$, we get $\dim(\langle \nu_d(C \cup L) \rangle) = 3d-1$, $\dim(\langle \nu_d(C) \rangle) = 2d$ and $\dim(\langle R \rangle) = d$. Hence Grassmann's formula gives $\langle \nu_d(C) \rangle \cap \langle R \rangle = \langle Z \rangle$. Fix $E \subset L$ such that $\{P\} = \langle Z \rangle \cap \langle \nu_d(E) \rangle$ (the " Moreover " part of Proposition 6). Since $\nu_d(E) \subset R$, P is the only point in the intersection of $\langle \nu_d(B) \rangle \subset \langle \nu_d(C) \rangle$ and $\langle \nu_d(E)$. Hence $\alpha(X,P) \le 3d$. Now assume $a := \alpha(X,P) < 3d$ and take $S = S_1 \cup \cdots \cup S_k \subset \mathbb{P}^m$ such that $\sharp(S) = a$ and $\{P\} = \cap_{i=1}^k \langle \nu_d(S_i) \rangle$. We proved that $\sharp(S_i) \ge d$ for all i. Hence $k = 2$, $2d \le a \le 3d-1$ and $d \le \sharp(S_i) \le 2d-1$ for all i.

Claim: Take a finite set $E \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(E) \rangle$, $P \notin \langle E' \rangle$ for any $E' \subsetneq E$, $E \ne A$, and $\deg(E) \le 2d-1$. Then $E \subset L$.

Proof of the Claim: Since $P \in \langle Z \rangle$, Lemma 3 and [8], Lemma 34, give the existence of a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (E \cup A)) \ge d+2$. First we will check that $E \subset D$ and then we will see that $D = L$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing D. Since E is reduced, A is curvilinear and H is general, we have $H \cap (A \cup E) = D \cap (A \cup E)$. Let $\mathrm{Res}_H(A \cup E)$ denote the residual scheme of $A \cup E$ with respect to H, i.e. the closed subscheme of $\mathbb{P}^m$ with $\mathcal{I}_{A \cup E} : \mathcal{I}_H$ as its ideal sheaf. Since $\deg(\mathrm{Res}_H(A \cup E)) = \deg(A \cup E) - \deg((A \cup E) \cap H)) \le d$, we have $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_H(A \cup E)}(d-1)) = 0$. Since A is connected and not reduced, [6], Lemma 4, gives $A \cup E \subset H$. Since this is true for a general H containing D, we get $E \subset D$. We also get $A \subset D$ and hence $D = L$.

Apply the Claim first to S_1 and then to S_2. We get $S \subset L$. Hence $\alpha(X,P) = \alpha(R,P) = d^2$, a contradiction. $\square$

Remark 5. Fix a linear subspace $U \subsetneq \mathbb{P}^m$ and take $P \in \langle \nu_d(U)\rangle$. We have $r_{X_{m,d}}(P) = r_{\nu_d(U)}(P)$ ([21], Proposition 3.1) and every $S \subset X$ evincing $r_X(P)$ is contained in $\nu_d(U)$ ([19], Exercise 3.2.2.2). Part (b) of Theorem 4 shows that sometimes $ir_X(P) < ir_{\nu_d(U)}(P)$.

Theorem 5. *Assume $m \ge 2$ and $d \ge 5$. Fix a finite set $A \subset \mathbb{P}^m$ such that $\sharp(A) = 3$. Set $X := X_{m,d}$ and $n := \binom{m+d}{m} - 1$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$.*

(a) Assume that A is contained in a line. Then $r_X(P) = z_X(P) = 3$, $ir_X(P) = iz_X(P) = d-1$ and $\alpha(X,P) = \gamma(X,P) = d+2$.

(b) Assume that A is not contained in a line. Then $r_X(P) = z_X(P) = 3$ and $\alpha(X,P) = 2d+2$.

Proof. Since $\beta(X) \ge 5$, $\nu_d(A)$ is the only subscheme of X with degree ≤ 3 whose linear span contains P. Hence $r_X(P) = z_X(P) = 3$. Since $\beta(X) = d+2$, Lemma 3 also gives $ir_X(P) \ge iz_X(P) \ge d-1$ and $\alpha(X,P) \ge \gamma(X,P) \ge d+2$.

First assume the existence of a line $L \subset \mathbb{P}^m$ such that $A \subset L$. Set $R := \nu_d(L)$. Since $P \in \langle R\rangle$, Proposition 7 gives $ir_X(P) \le ir_R(P) = d-1$, $iz_X(P) \le iz_R(P) = d-1$, $\alpha(X,P) \le \alpha(R,P) = d+2$ and $\gamma(X,P) \le \gamma(R,P) = d+2$, concluding the proof of part (a).

Now assume that A is not contained in a line. Write $A = \{O_1, O_2, O_3\}$. Fix $i \in \{1,2,3\}$ and set $\{j,h\} := \{1,2,3\} \setminus \{i\}$. Set $L_i := \langle \{O_j, O_h\}\rangle \subset \mathbb{P}^m$. Since $P \in \langle \nu_d(A)\rangle$ and $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$, the set $\langle\{P, \nu_d(O_i)\}\rangle \cap \langle\{\nu_d(O_h), \nu_d(O_j)\}\rangle$ is a single point, P_i. Notice that $P_i \in \langle \nu_d(L_i)\rangle$ and that $r_{\nu_d(L_i)}(P_i) = 2$. The " Moreover " part of Proposition 5 gives the existence of a set $E_i \subset L_i$ such that $\sharp(S_i) = d$ and $\{P_i\} = \langle\{\nu_d(O_h), \nu_d(O_j)\}\rangle \cap \langle \nu_d(E_i)\rangle$. Hence $\langle \nu_d(A)\rangle \cap \langle \nu_d(\{O_i\} \cup E_i)\rangle$ is the line $\langle \{\nu_d(O_i), P_i\}\rangle$. Taking the intersection of two of these lines we get $ir_X(P) \le d+1$ and $\alpha(X,P) \le 2d+2$. Since $r_X(P) = d+1$ (proof of this case in [8], Theorem 37), we get $ir_X(P) = d+1$. Lemma 3 also gives $iz_X(P) \ge d+1$ and that for each subscheme $W \subset \mathbb{P}^m$ with $\deg(W) \le d+1$ and $P \in \langle W\rangle$ we have $W \supseteq A$. Hence $iz_X(P) = d+1$. Assume $a := \alpha(X,P) \le 2d+1$ and take $S = S_1 \cup \cdots \cup S_k$ with $\{P\} = \cap_{i=1}^k \langle \nu_d(S_i)\rangle$ and $\sharp(S_1) + \cdots + \sharp(S_k) = a$. Since $a \le 2d+1$ and each subscheme $W \subset \mathbb{P}^m$ with $\deg(W) \le d+1$ and $P \in \langle W\rangle$ contains A, we get $k = 2$ and that one of the sets S_i is just A. Since $P \in \langle S_1\rangle \cap \langle S_2\rangle$, $P \notin \langle U\rangle$ for any $U \subsetneq S_i$, $i = 1,2$, and $\sharp(S_1 \cup S_2) \le 2d+1$, there is a line $D \subset \mathbb{P}^m$ such that $\sharp(D \cap (S_1 \cup S_2)) \ge d+2$ and $S_1 \setminus S_1 \cap D = S_2 \setminus S_2 \cap D$ ([6], Lemma 4). Since $S_1 \cap S_2 = \emptyset$, we get $S_1 \cup S_2 \subset D$. Since A is not contained in a line and $A = S_i$ for some i, we get a contradiction. □

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