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EIGENVALUES OF COMPOSITION COMBINED WITH DIFFERENTIATION

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ABSTRACT. Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane. Such a map induces through composition the linear composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$. The eigenvalues and the spectrum of such an operator acting on different spaces of analytic functions have been investigated in several articles, see e.g. [1], [8], [16], [28] and [29]. In this article we continue this line of research by combining the composition operator with the differentiation $D : H(\mathbb{D}) \to H(\mathbb{D}),\ f \to f'$. Then we obtain two linear operators $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \phi'(f' \circ \phi)$ and $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f' \circ \phi$. Now, we calculate the eigenvalues of the operators DC_ϕ and $C_\phi D$.

1. INTRODUCTION

Let $H(\mathbb{D})$ denote the class of all analytic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. In this article we consider an analytic self-map ϕ of $\mathbb{D}$. First, we consider the differentiation operator D given by

$$D : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f'.$$

Then we combine this with the composition operator

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}), f \mapsto f \circ \phi$$

to obtain the differentiation followed by composition

$$C_\phi D : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f' \circ \phi$$

and the composition followed by differentiation

$$DC_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \phi'(f' \circ \phi).$$

Obviously, the operators $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f' \circ \phi$ and $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \phi'(f' \circ \phi)$ are well-defined and bounded. The study of composition operators has quite a long and rich history. Among other reasons this comes from the fact that composition operators link operator theory with complex analysis. A very good introduction to the theory of composition operators is given in the excellent monographs by Shapiro [26] and Cowen and MacCluer [14]. Composition operators have been studied by many authors on various spaces of holomorphic functions, see e.g. [5], [6], [7], [9], [10], [11], [15], [17], [19], [22], [23], [24] and the references therein. Since the literature is growing steadily this can only be a sample of articles. The spectrum of the composition operator C_ϕ acting on various spaces has been determined by several authors, see e.g. the articles

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[1], [8], [16], [28] and [29]. In this article we continue this line of research by calculating the eigenvalues of the operators $C_\phi D$ and DC_ϕ. To do this we consider analytic self-maps of $\mathbb{D}$ that are not conformal automorphisms and have a fixed point $a \in \mathbb{D}$. For the study of both operators we need to consider the following two cases:

(a) a is an attracting fixed point of ϕ: In this case it turns out that both operators do not have any eigenvalues.
(b) a is a super-attracting fixed point of ϕ: Here both operators also show the same behavior, i.e. in case that $\phi(z) = z^2$ both operators have the eigenvalue 2 and in all the other cases both operators have no eigenvalues.

2. Results

We start this section with the introduction of the setting we are working in. In this article we are mainly interested in analytic self-maps of $\mathbb{D}$ that are not conformal automorphisms of $\mathbb{D}$ and have a fixed point $a \in \mathbb{D}$. We distinguish the following cases:

(1) a is an *attracting* fixed point of ϕ, i.e. $\phi'(a) \neq 0$. Model maps are functions $f(z) = \lambda z$ for $z \in \mathbb{D}$ with $|\lambda| < 1$.
One can change variables analytically in a neighborhood of a and conjugate ϕ to the map $f(z) = \lambda z$ for $\lambda = \phi'(a)$, for details see e.g. [25]. Originally, this was shown by Koenigs in [18]
(2) a is a *super-attracting* fixed point of ϕ, i.e. $\phi'(a) = 0$. In this case model maps are given by $\phi(z) = z^n$, $n \geq 2$. Again we can change variables analytically in a neighborhood of a and conjugate ϕ to the map $\phi(z) = z^n$ for some $n \geq 2$. The proof of this fact goes back to Böttcher [4].

2.1. Differentiation followed by composition.

We start with investigating the behavior of the operator

$$C_\phi D : H(\mathbb{D}) \to H(\mathbb{D}).$$

Then, we have the following:

Theorem 1. *Suppose that ϕ is a holomorphic self-map of $\mathbb{D}$ with fixed point 0. Moreover, we assume that $C_\phi D f = \lambda f$ holds for some $\lambda \in \mathbb{C}$ and a function f of the type*

$$f(z) = \sum_{l=n}^{\infty} a_l z^l \in H(\mathbb{D}),$$

for some $n \geq 2$. Then, $\lambda = (n-1)\phi'(0)^{n-2}\phi''(0)$.

Proof. Obviously, the assumption yields

$$f'(\phi(z)) = \lambda f(z), \text{ for every } z \in \mathbb{D} \text{ and some } \lambda \in \mathbb{C}.$$

Since $f(z) = \sum_{l=n}^{\infty} a_l z^l$ and therefore $f'(z) = \sum_{l=n+1}^{\infty} l a_l z^{l-1}$ we arrive at the following equation

$$\begin{aligned} \lambda = \frac{f'(\phi(z))}{f(z)} &= \frac{\phi(z)^{n-1}}{z^n} \cdot \frac{n a_n + a_{n+1}\phi(z) + \dots}{a_n + a_{n+1} z + \dots} \\ &= \left(\frac{\phi(z)}{z}\right)^{n-1} \frac{1}{z} \cdot \frac{n a_n + a_{n+1}\phi(z) + \dots}{a_n + a_{n+1} z + \dots}. \end{aligned}$$

Now, letting $z \to 0$ and applying the rule of L'Hôpital we have

$$\lambda = (n-1)\phi'(0)^{n-2}\phi''(0),$$

as desired. □

Next, we study the situation in case that f is either a constant function or a function of the form $f(z) = cz + d$ for every $z \in \mathbb{D}$ and some constants $c, d \in \mathbb{C}$.

Remark 1. (a) *We assume that $f(z) = c$ for every $z \in \mathbb{D}$, $c \neq 0$, and that the equation $C_\phi Df(z) = \lambda f(z)$ holds for every $z \in \mathbb{D}$ and some $\lambda \in \mathbb{C}$. Obviously this yields $0 = \lambda c$. Since $c \neq 0$ this means that λ must be equal to zero.*

(b) *Now, we suppose that f is a holomorphic function of the form $f(z) = cz + d$ for every $z \in \mathbb{D}$ and some $c, d \in \mathbb{C}$. Moreover let $C_\phi Df(z) = \lambda f(z)$ for every $z \in \mathbb{D}$ and some $\lambda \in \mathbb{C}$. Then $f'(\phi(z)) = \lambda f(z)$ for every $z \in \mathbb{D}$ is equivalent with $c = \lambda cz + \lambda d$. Hence λ must be equal to zero.*

Corollary 1. *Operators $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$ induced by a rotation ϕ, i.e. by a map ϕ of the form $\phi(z) = e^{i\Theta\pi}z$, $z \in \mathbb{D}$, where $\Theta \in [0, 2)$ is fixed, do not have any eigenvalues.*

In the following we will determine the eigenvalues of the operator $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$ induced by a symbol of the form $\phi(z) = \nu z$ for every $z \in \mathbb{D}$, where $\nu \in \mathbb{C}$, $|\nu| < 1$, or a symbol of the form $\phi(z) = z^n$ for every $z \in \mathbb{D}$ and some $n \geq 2$. We start with the first case.

Corollary 2. *Let ϕ be of the form $\phi(z) = \nu z$ for some $\nu \in \mathbb{C}$ with $|\nu| < 1$. Then the induced operator $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$ has no eigenvalues.*

We can prove this corollary in another way by using a power series argument, which we will do below.

Theorem 2. *Let ϕ be of the form $\phi(z) = \nu z$, for every $z \in \mathbb{D}$, and some $\nu \in \mathbb{C}$ with $|\nu| < 1$. Then the operator*

$$C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$$

has no eigenvalues.

Proof. We show this by contradiction and assume that we can find an eigenvalue $\mu \in \mathbb{C}$, $\mu \neq 0$. Then there exists an eigenfunction f given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for every } z \in \mathbb{D},$$

where $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$, are suitable coefficients. Now, we obtain the following derivative

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

and

$$[C_\phi D](f(z)) = \sum_{n=1}^{\infty} n a_n \nu^{n-1} z^{n-1}.$$

Hence, $[C_\phi D](f(z)) = \mu f(z)$ for every $z \in \mathbb{D}$ holds if and only if

$$\sum_{n=1}^{\infty} a_n n \nu^{n-1} z^{n-1} = \sum_{n=0}^{\infty} \mu a_n z^n.$$

Now, we compare the coefficients. If $a_0 = 0$, then we obtain successively, that $a_n = 0$ for every $n \in \mathbb{N}$. In this case we are done. Thus, without loss of generality, we may assume that $a_0 \neq 0$. Next, we have that

$$a_1 = \mu a_0 \text{ which is equivalent with } \mu = \frac{a_1}{a_0}.$$

Moreover, we obtain $2a_2\nu = \mu a_1 = \mu^2 a_0$ and hence

$$a_2 = \frac{\mu a_1}{2\nu} = \frac{\mu^2 a_0}{2\nu}.$$

For every $n \geq 3$ we have the following formula

$$a_n = \frac{\mu^n a_0}{n!\nu^{n(n-2)-\sum_{k=2}^{n-2}(n-k)}}. \tag{2.1}$$

We show this inductively. For $n = 3$ a comparison of coefficients yields

$$3a_3\nu^2 = \mu a_2 = \frac{\mu^2 a_1}{2\nu} \iff a_3 = \frac{\mu^2 a_1}{6\nu^3} = \frac{\mu^3 a_0}{6\nu^3}.$$

Next, we assume that (2.1) is satisfied for some $n \in \mathbb{N}$. Again, by comparison of coefficients we get

$$(n+1)a_{n+1}\nu^n = \mu a_n = \frac{\mu^{n+1} a_0}{n!\nu^{n(n-2)-\sum_{k=2}^{n-2}(n-k)}}$$

is equivalent to

$$a_{n+1} = \frac{\mu^{n+1} a_0}{(n+1)!\nu^{(n-2)n-\sum_{k=2}^{n-2}(n-k)+n}}.$$

Easy calculations show

$$n(n-2) - \sum_{k=2}^{n-2}(n-k) + n = (n-1)(n+1) - \sum_{k=2}^{n-1}(n+1-k)$$

is equivalent to $-n+1 = -n+1$. Hence, the claim follows.

Next, we compute the radius of convergence of the power series generated by the coefficients we got above and arrive at

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\mu}{(n+1)\nu^n}\right| = \infty,$$

since $|\nu| < 1$. Hence the radius is 0 and f is not a holomorphic function in $\mathbb{D}$ which is a contradiction. Finally, the claim follows. □

Next, we turn our attention to symbols of the form $\phi(z) = z^n$, for every $z \in \mathbb{D}$ and some $n \geq 2$. First, again we obtain a corollary of Theorem 1 and Remark 1.

Corollary 3. *Let ϕ be of the form $\phi(z) = z^n$ for every $z \in \mathbb{D}$ and some $n \geq 2$. If $n = 2$, only $\lambda = 2$ may be an eigenvalue of the operator $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$. In $n \geq 3$ the operator $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$ does not have any eigenvalues.*

Using the same methods as in Theorem 2 we arrive at:

Theorem 3. *Let ϕ be of the form $\phi(z) = z^n$ for every $z \in \mathbb{D}$ with $n \geq 2$. Then, in case of $n = 2$, $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$ has the unique eigenvalue $\mu = 2$, while for $n \geq 3$ the operator has no eigenvalues.*

Proof. First, we treat the case $n = 2$. Obviously, with the function $f(z) = z^2$, we get

$$[C_\phi D](f(z)) = 2z^2 = 2f(z) \text{ for every } z \in \mathbb{D}.$$

To show that μ is unique, we assume that there is another eigenvalue ν. In that case there must be an eigenfunction $f \in H(\mathbb{D})$ which can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^n \text{ with some coefficients } a_k \in \mathbb{C}.$$

Hence

$$f'(z) = \sum_{n=1}^{\infty} a_k z^{k-1} \quad \text{and} \quad f'(\phi(z)) = f'(z^2) = \sum_{k=1}^{\infty} a_k k z^{2k-2}.$$

Then $[C_\phi D](f(z)) = \nu f(z)$ holds for every $z \in \mathbb{D}$ if and only if

$$\sum_{k=1}^{\infty} a_k k z^{2k-2} = \nu \sum_{k=0}^{\infty} a_k z^k$$

which is equivalent to

$$a_1 + 2a_2 z^2 + 3a_3 z^4 + 4a_4 z^6 + \cdots = \nu a_0 + \nu a_1 z + \nu a_2 z^2 + \ldots$$

Then a comparison of coefficients yields $a_1 = \nu a_0$, $a_1 = 0$ and hence $a_0 = 0$, $2a_2 = \nu a_2$ and $a_k = 0$ for every $k \geq 3$. Thus, a_2 either is 0 but then $f \equiv 0$ on $\mathbb{D}$ which is a contradiction or $\nu = 2$. Hence the claim follows. Next, we consider the case $n \geq 3$. Again we show this indirectly and assume that there is an eigenvalue $\mu \in \mathbb{C}$, $\mu \neq 0$. Then, since, the eigenfunction f must be an element of $H(\mathbb{D})$ we get that

$$[C_\phi D](f(z)) = \mu f(z) \text{ for every } z \in \mathbb{D}$$

is equivalent to

$$\sum_{k=1}^{\infty} a_k k z^{nk-n} = \sum_{k=0}^{\infty} \mu a_k z^k \text{ for every } z \in \mathbb{D}.$$

But this is equivalent to

$$a_1 + 2a_2 z^n + 3a_3 z^{2n} + 4a_4 z^{3n} + \cdots = \mu a_0 + \mu a_1 z + \mu a_2 z^2 + \ldots$$

Hence a comparison of coefficients yields that $a_k = 0$ for every $k \in \mathbb{N}_0$. Thus, the claim follows. □

2.2. **Composition followed by differentiation.** In this section we study composition followed by differentiation $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$. We use the same methods and ideas as in the previous section but for the reader's benefit we give the full proofs.

Theorem 4. *Suppose that ϕ is a holomorphic self-map of $\mathbb{D}$ with fixed point 0. Moreover, we assume that $DC_\phi f = \lambda f$ holds for some $\lambda \in \mathbb{C}$ and a function f of the type*

$$f(z) = \sum_{l=n}^{\infty} a_l z^l \in H(\mathbb{D}),$$

with $n \geq 2$. Then, $\lambda = n\phi'(0)^{n-1}\phi''(0)$.

Proof. By assumption we have that $f'(\phi(z))\phi'(z) = \lambda f(z)$ for every $z \in \mathbb{D}$ and some $\lambda \in \mathbb{C}$. Since $f(z) = \sum_{l=n}^{\infty} a_l z^l$ and therefore $f'(z) = \sum_{l=n}^{\infty} l a_l z^{l-1}$ we arrive at the following equation

$$\lambda = \frac{f'(\phi(z))\phi'(z)}{f(z)} = \phi'(z)\frac{\phi(z)^{n-1}}{z^n}\frac{na_n + a_{n+1}\phi(z) + \ldots}{a_n + a_{n+1}z + \ldots} =$$
$$= \phi'(z)\left(\frac{\phi(z)}{z}\right)^{n-1}\frac{1}{z}\frac{na_n + a_{n+1}\phi(z) + \ldots}{a_n + a_{n+1}z + \ldots}$$

Now, letting $z \to 0$ we get

$$\lambda = n\phi'(0)^{n-1}\phi''(0),$$

as desired. □

It remains to study the case when f is either a constant function or a function of the form $f(z) = cz + d$ with $c, d \in \mathbb{C}$.

Remark 2. (1) *We assume that $f(z) = c$ for every $z \in \mathbb{D}$, $c \neq 0$ and that the equation $DC_\phi f(z) = \lambda f(z)$ holds for every $z \in \mathbb{D}$ and some $\lambda \in \mathbb{C}$. Then, obviously, $\phi'(z)f'(\phi(z)) = \lambda f(z)$ for every $z \in \mathbb{D}$ is equivalent with $0 = \lambda c$. Since $c \neq 0$, λ must be equal to zero.*

(2) *Next, we suppose that the equation $DC_\phi f(z) = \lambda f(z)$ holds for every $z \in \mathbb{D}$, some $\lambda \in \mathbb{C}$ and a function $f(z) = cz + d$ for every $z \in \mathbb{D}$, $c, d \in \mathbb{C}$. Then $\phi'(z)f'(\phi(z)) = \lambda f(z)$ holds if and only if $\phi'(z)c = \lambda(cz + d) = \lambda cz + \lambda d$. But this is satisfied if and only if $\phi'(z) = \lambda z + \frac{\lambda}{c}d$ for some $\lambda \in \mathbb{C}$. Hence, this is always fulfilled if ϕ is given by $\phi(z) = \lambda z^2 + \frac{\lambda}{c}dz + k$ with some constant $k \in \mathbb{C}$.*

Corollary 4. *Operators $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ induced by a rotation ϕ do not have any eigenvalues.*

In the following we will determine the eigenvalues of the operator $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ induced by a symbol of the form $\phi(z) = \nu z$ for every $z \in \mathbb{D}$, where $\nu \in \mathbb{C}$, $|\nu| < 1$, or a symbol of the form $\phi(z) = z^n$ for every $z \in \mathbb{D}$ and some $n \geq 2$. We start with the first case.

Corollary 5. *Let ϕ be of the form $\phi(z) = \nu z$ for some $\nu \in \mathbb{C}$ with $|\nu| < 1$. Then the induced operator $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ has no eigenvalues.*

We can prove this corollary in another way by using a power series argument, which we will do below.

Theorem 5. *Let ϕ be a holomorphic self-map of $\mathbb{D}$ that has an attracting fixed point $a \in \mathbb{D}$, i.e. we assume ϕ to be of the form $\phi(z) = \nu z$ for some $\nu \in \mathbb{C}$ with $|\nu| < 1$ and every $z \in \mathbb{D}$. Then the operator $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ has no eigenvalues.*

Proof. We show this indirectly and assume to the contrary that we can find an eigenvalue $\mu \in \mathbb{C}$, $\mu \neq 0$. Then the eigenfunction can be written in the following way

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with suitable coefficients } a_n \in \mathbb{C},\ n \in \mathbb{N}_0.$$

Now, the derivative is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \text{ and}$$

$$[DC_\phi](f(z)) = \sum_{n=1}^{\infty} n a_n \nu^n z^{n-1} = \phi'(z)f'(\phi(z)) = \nu \sum_{n=1}^{\infty} n a_n \nu^{n-1} z^{n-1}.$$

This yields that the equation $[DC_\phi](f(z)) = \mu f(z)$ holds for every $z \in \mathbb{D}$ if and only if

$$\sum_{n=1}^{\infty} n a_n \nu^n z^{n-1} = \sum_{n=0}^{\infty} \mu a_n z^n.$$

If $a_0 = 0$, we obtain successively that $a_n = 0$ for every $n \in \mathbb{N}$. In this case we are done. Thus, w.l.o.g. we may assume that $a_0 \neq 0$. Next, we have that

$$a_1 \nu = \mu a_0 \Longleftrightarrow a_1 = \frac{\mu}{\nu} a_0.$$

Furthermore another comparison of coefficients yields

$$2a_2\nu^2 = \mu a_1 = \frac{\mu}{\nu}a_0 \Longleftrightarrow a_2 = \frac{\mu^2}{2\nu^3}a_0.$$

For every $n \geq 3$ the following formula holds

$$a_n = \frac{\mu^n a_0}{n!\nu^{(n+1)(n-1)-\sum_{k=2}^{n-1}(n+1-k)}}. \tag{2.2}$$

We prove this formula by induction. In case $n = 3$ the comparison of coefficients yields

$$3a_3\nu^3 = \mu a_2 = \frac{\mu^3}{2\nu^3}a_0 \Longleftrightarrow a_3 = \frac{\mu^3}{6\nu^6}a_0.$$

Now, we assume that (2.2) holds for some $n \in \mathbb{N}$, $n \geq 3$. We obtain

$$(n+1)a_{n+1}\nu^{n+1} = \mu a_n = \frac{\mu^{n+1}}{n!\nu^{(n+1)(n-1)-\sum_{k=2}^{n-1}(n+1-k)}}a_0,$$

which means that

$$a_{n+1} = \frac{\mu^{n+1}}{(n+1)!\nu^{(n+1)(n-1)-\sum_{k=2}^{n-1}(n+1-k)+n+1}}.$$

Now, easy calculations show

$$(n+1)(n-1) - \sum_{k=2}^{n-1}(n+1-k) + n + 1 = (n+2)n - \sum_{k=2}^{n}(n+2-k)$$
$$\Longleftrightarrow -n = -n$$

and the claim follows. □

Next, we turn our attention to symbols of the form $\phi(z) = z^n$ for every $z \in \mathbb{D}$ and some $n \geq 2$. First, again we obtain a corollary of Theorem 4 and Remark 2.

Corollary 6. *Let ϕ be of the form $\phi(z) = z^n$ for every $z \in \mathbb{D}$ and some $n \geq 2$. If $n = 2$, only $\lambda = 2$ may be an eigenvalue of the operator $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$. In $n \geq 3$ the operator $C_\phi D : H(\mathbb{D}) \to H(\mathbb{D})$ does not have any eigenvalues.*

Again, we can prove this using another method involving power series.

Theorem 6. *Let ϕ be an analytic self-map of $\mathbb{D}$ that is not a conformal automorphism and has a super-attracting fixed point $a \in \mathbb{D}$, i.e. we assume ϕ to be of type $\phi(z) = z^n$, $n \geq 2$. Then, in case $n = 2$, $DC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ has the unique eigenvalue $\mu = 2$, while for $n \geq 3$, the operator has no eigenvalues.*

Proof. First, we treat the case $n = 2$. Obviously, with the function $f(z) = z$ we have that

$$[DC_\phi](f(z)) = 2z = 2f(z) \text{ for every } z \in \mathbb{D} :$$

Hence $\mu = 2$ is an eigenvalue. To show that it is the only one, we assume to the contrary that there is another eigenvalue ν. In this case we can find an eigenfunction $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for every $z \in \mathbb{D}$. Then the derivative can be written as

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

This yields

$$f'(z^2) = f'(\phi(z)) = \sum_{k=1}^{\infty} a_k k z^{2k-2}$$

and thus we obtain

$$\phi'(z)f'(\phi(z)) = 2zf'(z^2) = 2\sum_{k=1}^{\infty} a_k k z^{2k-1}.$$

Then $[DC_\phi](f(z)) = \nu f(z)$ holds for every $z \in \mathbb{D}$ if and only if

$$2\sum_{k=1}^{\infty} a_k k z^{2k-1} = \nu \sum_{k=0}^{\infty} a_k z^k.$$

Hence a comparison of coefficients yields that either $\nu = 0$ or $\nu = 2$ and we are done. Next, we consider the case $n \geq 3$. Again, we show this indirectly and assume that there is an eigenvalue $\mu \in \mathbb{C}$, $\mu \neq 0$. Then, since the eigenfunction f must be an element of $H(\mathbb{D})$, $[DC_\phi](f(z)) = \mu f(z)$ holds for every $z \in \mathbb{D}$ if and only if

$$\sum_{k=1}^{\infty} a_k k n z^{nk-1} = \mu \sum_{k=0}^{\infty} a_k z^k$$

A comparison of coefficients yields that $a_k = 0$ for every $k \in \mathbb{N}_0$. Finally, the claim follows. $\square$

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